Lecture 20 - Oscillatory motion and pendula

What's important: (lecture takes more than an hour to complete)

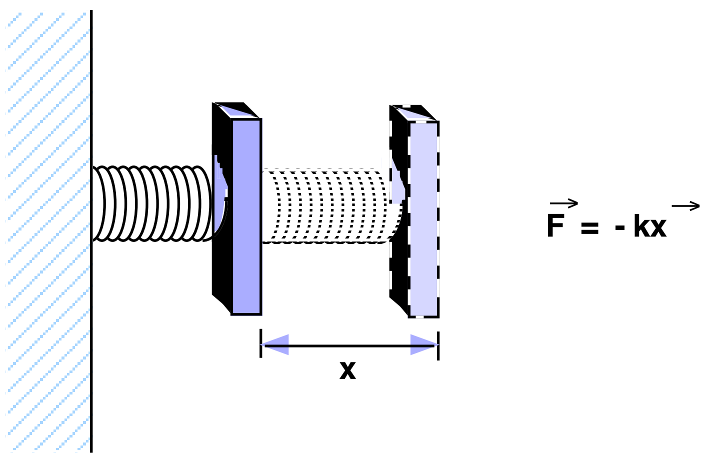
- simple harmonic motion
- oscillation period of a spring
- simple pendulum

Demonstrations:

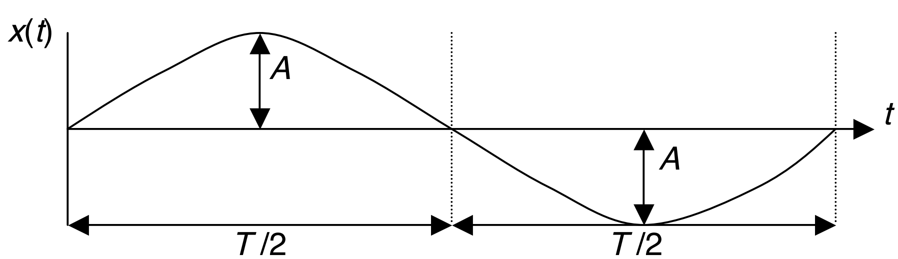
- mass on a spring
- projected motion
- simple pendulum
- Pasco track with computer data acquisition

Oscillatory Motion

We introduced Hooke's Law as an example of a force that increases linearly with the displacement from equilibrium:

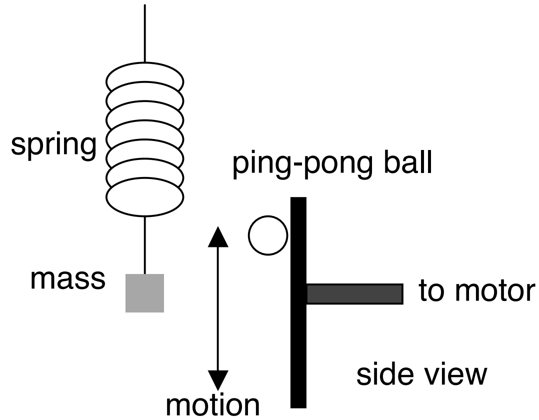


The motion of an object obeying Hooke's Law is referred to as simple harmonic motion (or SHM). The apparent form of this oscillatory motion can be seen by making a plot of $x(t)$:

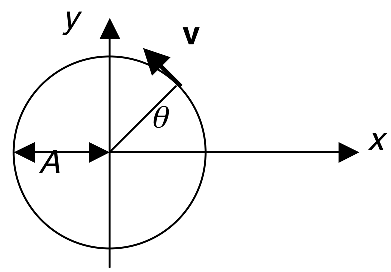


Drawn properly, this curve would look suspiciously like a sine or cosine function. Although this can be established directly with calculus, we must take a somewhat circuitous route here:, namely the projection of uniform circular motion on an axis.

Demo: mass on a spring compared to rotation of a ping-pong ball attached to a disk



Mathematically, the motion of the ball is:



The circle has a radius A and the object moves with a speed v_o. Starting at $\theta = 0$ when $t = 0$, the y-component is given by

$$y(t) = A \sin\theta.$$

But θ is a function of t, according to the usual

$$\theta = \omega t,$$

where ω is the angular speed, so we have

$$y(t) = A \sin \omega t. \qquad (1)$$

The projection of the velocity vector on the y-axis also varies with time according to

$$v_y = v_o \cos\theta.$$

Substituting $\theta = \omega t$ and $v_o = \omega A$ (which is just $v = \omega R$ for this situation), we have

$$v_y = \omega A \cos \omega t. \qquad (2)$$

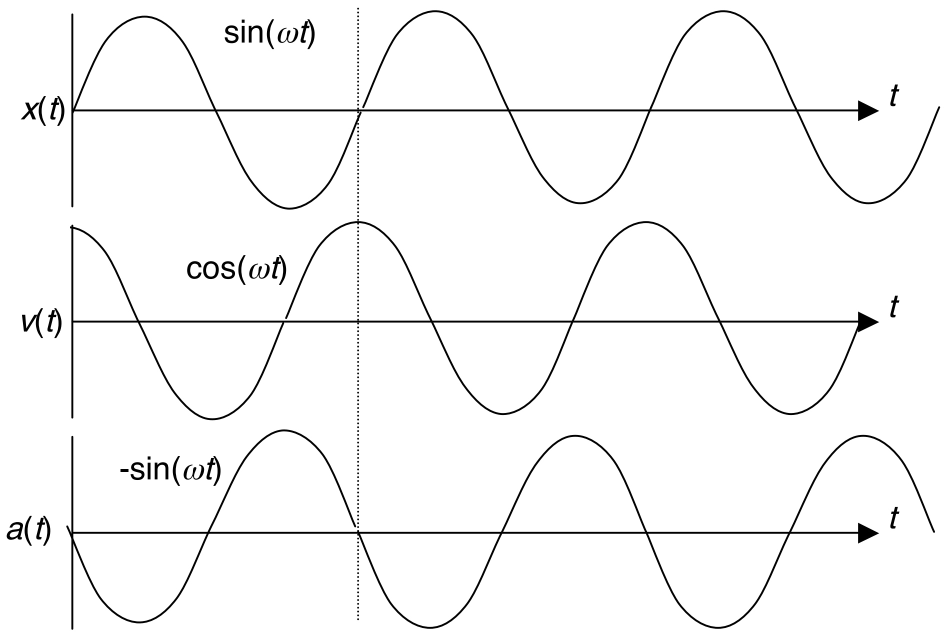
Lastly, the magnitude of the centripetal acceleration is

$$a = v_o^2/A = \omega^2 A$$

Again, the y component of this is

$$a_y = - \omega^2 A \sin \omega t. \qquad (3)$$

Demo: show the horizontal motion of a cart on a spring agrees with:



OK, so we've now found y, v_y and a_y for the projection, which we know is sinusoidal. The next step is to find the relation between a_y and y. This is easy. Start with (3) and group the terms as

$$a_y = -\omega^2 (A \sin \omega t) \qquad (4)$$

then substitute (1) into (4)

$$a_y(t) = -\omega^2 y(t). \qquad (5)$$

This is the relation between acceleration and position for the projection, valid for all times t. Does the spring obey this? YES, as can be seen from Newton's law:

$$F = ma = -kx$$

or

$$a = -(k/m)\, x \qquad (6)$$

In other words, the spring obeys sinusoidal motion, with an angular speed

$$\omega^2 = k/m$$

or

$$\omega = (k/m)^{1/2}. \qquad (7)$$

We can obtain the frequency and period for the motion through the usual substitutions

$$\omega = 2\pi f = 2\pi/T$$

so

$$T = 2\pi(m/k)^{1/2} \qquad \text{and} \qquad f = (1/\,2\pi) \cdot (k/m)^{1/2}.$$

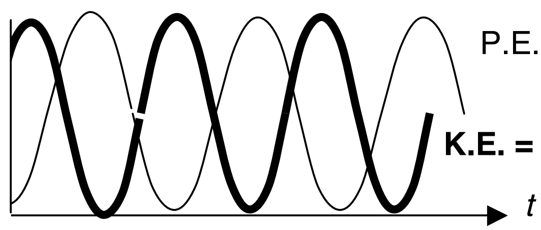
Just to emphasize, the characteristic relationship for simple harmonic motion is

$$a = -\omega^2 x$$

Demo: calculate k and measure m of the oscillating spring, and confirm the expression for period.

Energy Conservation in SHM

The kinetic and potential energy for a spring in simple harmonic motion looks like:



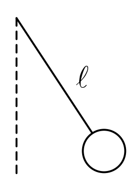
By explicitly calculating K and U with the expressions for x and v, one can show

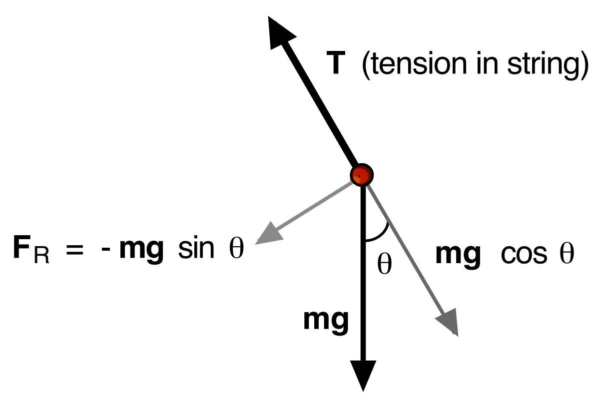
$$E = K + U = (1/2)\ kA^2 \cos^2 \omega t \ + \ (1/2)\ kA^2 \sin^2 \omega t$$
$$= (1/2)\ kA^2 (\cos^2 \omega t + \sin^2 \omega t)$$
$$= kA^2/2.$$

Since both k and A are constants, then so is E – total energy is conserved.

Simple Pendulum

Consider a mass m suspended by a massless string of length ℓ. When displaced from its equilibrium position, the mass is subject to a restoring force



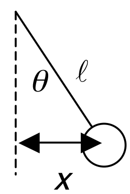


In a coordinate system which has one axis along the string,

$T = mg\cos\theta$ balanced

$F_R = -mg\sin\theta$ unbalanced

Let's work in linear coordinates, rather than angles, and use the distance *x* of the mass from the vertical, as in



Now, $\sin\theta = x/l$, so

$$F_R = -mg\,x/l$$

For θ small, F_R is roughly horizontal, as is *x*, so

$$F_R = ma = -mg\,x/l$$

or

$$a = -g\,x/l.$$

The relation between *a* and *x* is that of simple harmonic motion, so

$\omega = (g\,/\,l)^{1/2}$ or $T = 2\pi(l\,/g)^{1/2}$.

Note: *T* does *not* depend on the mass *m* or the amplitude *A*!!

Example What is the period of a pendulum 1.00 m long?

$$\mathbf{T} = 2\pi\sqrt{\frac{1.00}{9.81}} = 2.00_6 \text{ sec.}$$