Lecture 21 - Waves on a string

What's important:
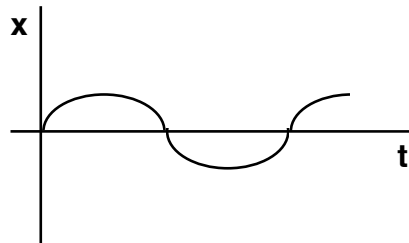•   speed of a wave on a string under tension
Demonstrations:
•   slinky toy to show transverse and longitudinal waves
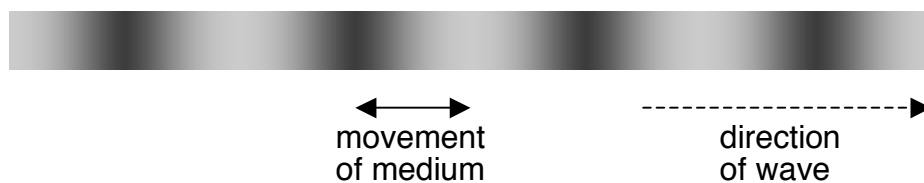
Waves on a String

We introduced oscillations in terms of motion of single objects subject to a restoring force. We saw that the resulting motion was wavelike (i.e. a sine or cosine function) as a function of time:

Now let's consider the oscillatory motion of an extended, deformable object, namely a string. We observe both transverse waves
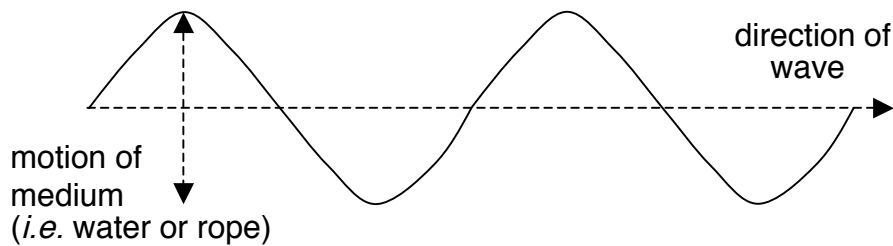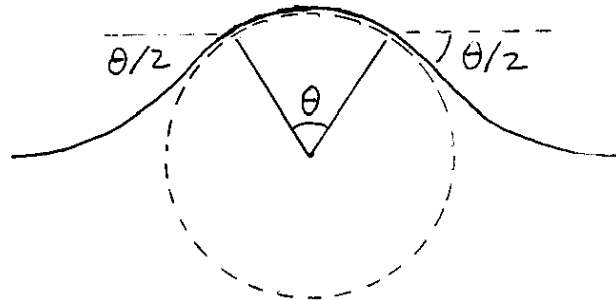

and longitudinal waves

(compression waves, like sound or seismic waves)

On a string, the frequency of a wave depends upon its velocity. We can find the form of the relationship by considering a wave pulse. Approximate the segment at the top of the wave as part of a circle with arc length $\Delta s = R\theta$, as in the diagram

where the relationships among the angles is obtained from simple trigonometry
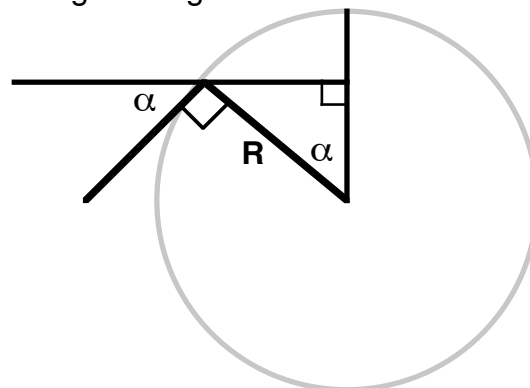
The forces on the segment Δs are from the tension **F** at each end of the segment
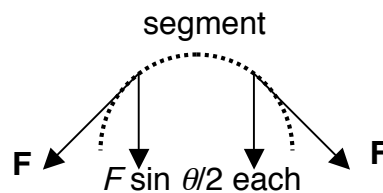
where **F** is used to represent tension, rather than the more obvious **T**, since **T** is already used for the oscillation period. The components of **F** acting to restore the segment to its equilibrium position are $F \sin\theta/2$ at each end. Suppose that θ is small, so that we can make the small angle approximation $\sin\theta/2 \sim \theta/2$. The net force that the segment experiences in the transverse direction is then

$$2\,F \sin\,\theta/2 \sim F\theta \qquad\qquad (1)$$

Now, θ is related to the arc length Δs by

$\theta = \Delta s\,/\,R$

so that the expression for the restoring force (1) can be rewritten as

$$F \, \Delta s \, / R.$$

This force gives the string element an acceleration in the transverse direction, which can be thought of as follows. Imagine that we observe the wave by travelling along beside it with a velocity v. If we look at the segment at the top of the wave, then it appears to experience a centripetal acceleration a_c, given by the usual expression

$$a_c = v^2 \, / R.$$
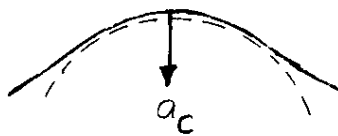
Thus, the force $F \, \Delta s \, / R$ gives the segment of mass $\mu \, \Delta s$ (where μ is the mass per unit length of string) an acceleration $v^2 \, / R$. Substituting into Newton's Second Law
$$[force] = [mass] \cdot [acceleration]$$

$$[force] = F \, \Delta s \, / R \qquad\qquad [mass] = \mu \, \Delta s \qquad\qquad [acceleration] = v^2 \, / R$$

we have
$$F \, \Delta s \, / R = (\mu \, \Delta s) \, (v^2 \, / R)$$
$$\Rightarrow \quad F = \mu v^2$$

or
$$v = (F \, / \, \mu)^{1/2} \tag{2}$$

Equation (2) gives the velocity of the wave in terms of the tension in the string.

Note:
(i) both R and θ have disappeared from Eq. (2).
(ii) the units work out as expected:
$$[force] \sim kg \, m^2 \, / \, s^2$$
$$[\mu] \sim kg \, / \, m$$
$$\Rightarrow [\, (force \, / \, \mu)^{1/2} \,] \sim (m^2 \, / \, s^2)^{1/2} = m \, / \, s.$$