Lecture 22 - Standing and travelling waves

What's important:

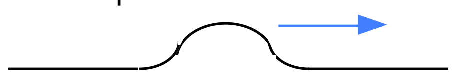
- vibrational frequency as a function of tension, mass
- travelling waves

Demonstrations:

- retort stand, elastic band, 0.5 kg mass, monochord

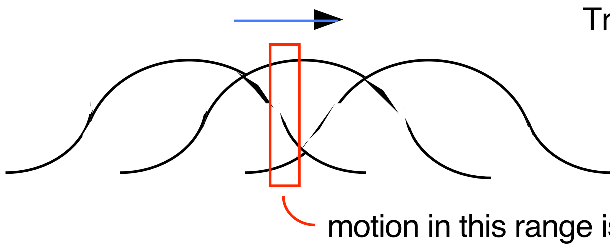
Standing and Travelling Waves

If we pluck a long string and release it, we can see a travelling wave move off from the place where the wave was plucked.



At any given point on the string, the movement of the string transverse to the direction of motion is SHM:

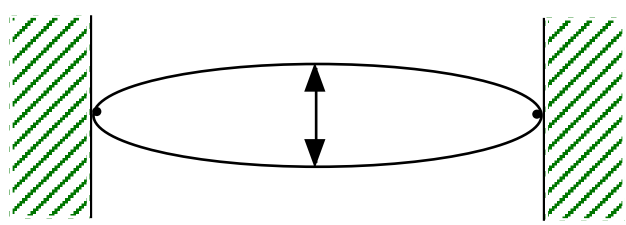
Travelling wave



motion in this range is SHM, with $\mathbf{v} = \sqrt{\frac{\mathbf{F}}{\mu}}$

Now, if the string is fixed at each end, then there is no motion at the end points, and the wave is called a *standing wave*:

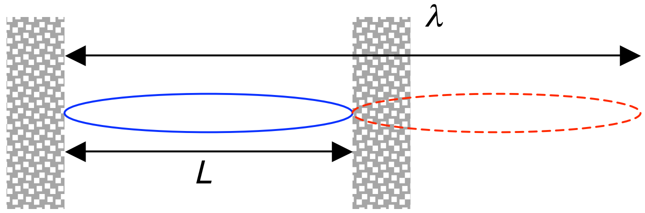
Standing wave



Any element of a standing wave executes SHM corresponding to a velocity of $(\mathbf{F}/\mu)^{1/2}$, even though the wave does not appear to move along the string. Because the wave velocity may not be apparent, it is sometimes more intuitive to think in terms of **f** and λ [frequency and wavelength].

$$\Rightarrow \quad \mathbf{v} = \mathbf{f}\lambda \qquad\qquad \Rightarrow \quad \mathbf{f} = \frac{\mathbf{v}}{\lambda} = \sqrt{\frac{\mathbf{F}}{\mu}} \cdot \frac{1}{\lambda}$$

The wavelength of the longest standing wave allowed by fixed walls separated by a length **L** is $f = v / 2L$:

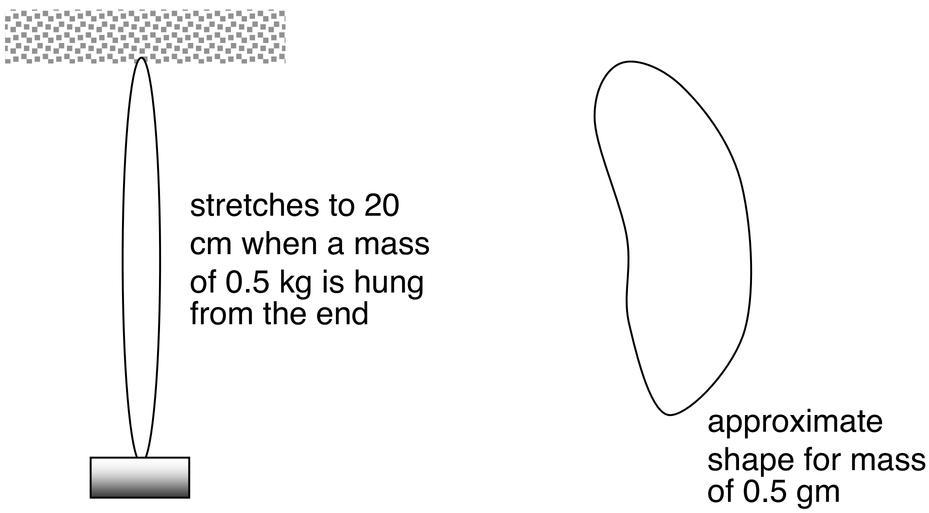


In general, $f = v / 2L,\ 2v / 2L,\ 3v / 2L, \ldots nv / 2L$ where $n = 1, 2, 3, \ldots$

Hence, the *lowest* frequency (largest λ) of a standing wave is

$$\mathbf{f} = \sqrt{\frac{\mathbf{F}}{\mu}}\ \frac{1}{2\mathbf{L}}$$

Demonstration Hang a mass from an elastic band and pluck it. The following results were obtained with an “average” elastic band of total mass 0.5 gm



For *one* side (of a two “sided” elastic)

$$\mu = \text{mass per unit length} = \frac{0.5 \times 10^{-3} / 2}{0.20}$$

0.5 g for total band

20 cm

$$\simeq 10^{-3}\ \text{kg / m}$$

$$\mathbf{F} = \frac{0.5 \times 9.8}{2}$$ ← 0.5 kg mass is supported by two sides

$$= 2.5 \text{ N}$$

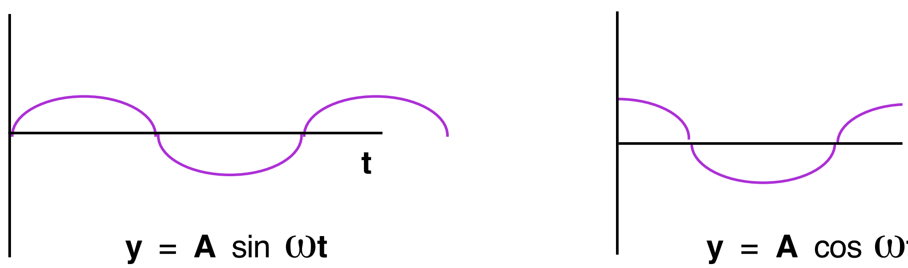
$$\Rightarrow \quad \mathbf{v} = \sqrt{\frac{\mathbf{F}}{\mu}} = \sqrt{\frac{2.5}{10^{-3}}} = \sqrt{2500} = 50 \text{ m / s}$$

The corresponding frequency is $f = v/\lambda = v / 2L = 50 / (2 \cdot 0.2) = 125 \text{ s}^{-1}$.
This compares to middle C at 256 Hz.

Travelling waves

When we solved the equations for simple harmonic motion, we said that either a sine or a cosine function was a valid solution. One chooses the solution according to the desired value of the displacement **y** at **t** = 0. (Note: we have changed notation and used **y** as displacement, for reasons that become obvious in the next paragraph).

y(**t**) = displacement



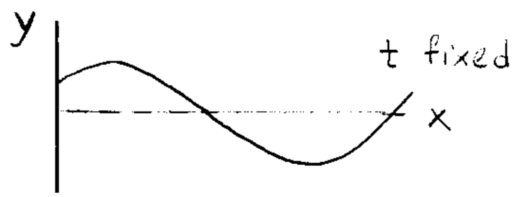
$\mathbf{y} = \mathbf{A} \sin \omega \mathbf{t}$ $\qquad$ $\mathbf{y} = \mathbf{A} \cos \omega \mathbf{t}$

These solutions have |**y**(**t**=0)| = 0 or **A**. A more general solution that allows for the complete range of **y**(**t**=0) is

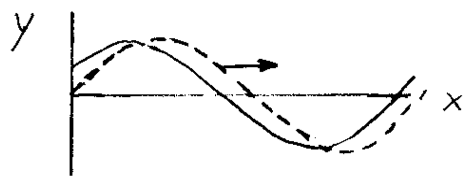
$$\mathbf{y}(\mathbf{t}) = \mathbf{A} \sin(\omega \mathbf{t} + \delta) = \mathbf{A} \sin(2\pi \mathbf{t} / \mathbf{T} + \delta)$$

where **T** is the period and δ is called the phase angle. The presence of δ allows the wave pattern along the time-axis to be shifted.

When we have a travelling wave (as opposed to a spring oscillating in one dimension) the displacement **y** depends on both time and the position in space **x** at which the displacement is measured. At a fixed time $\mathbf{t}_1$, the displacement as a function of **x** is denoted by $\mathbf{y}(\mathbf{x},\mathbf{t}_1)$, and has the graphical representation:



At a later time $\mathbf{t}_2$, the wave has moved to the right, and the displacement $\mathbf{y}(\mathbf{x},\mathbf{t}_2)$ looks like



Let's write out mathematically what is contained in the diagrams. For a fixed value of **t**, the wave equation *as a function of x* has the form

$$\mathbf{y}(\mathbf{x}) = \mathbf{A} \sin (2\pi\mathbf{x}/\lambda) \qquad \lambda = \text{wavelength}$$

just like the time-dependence of the harmonic oscillator. The effect of the wave moving is to introduce a time-dependent phase factor $2\pi\mathbf{t} / \mathbf{T}$, which carries a minus sign if the motion is to the right. In other words, the expression for **y** must depend on both **x** and **t**. If $\mathbf{y}(\mathbf{x}=0, \mathbf{t}=0) = 0$, then the wave is described by:

$$\mathbf{y}(\mathbf{x,t}) = \mathbf{A} \sin (2\pi\mathbf{x}/\lambda - 2\pi\mathbf{t}/\mathbf{T}) \qquad \text{travels to right}$$

A wave travelling to the left has a very similar functional form, but with the opposite sign for the "time" argument:

$$\mathbf{y}(\mathbf{x,t}) = \mathbf{A} \sin (2\pi\mathbf{x}/\lambda + 2\pi\mathbf{t}/\mathbf{T}) \qquad \text{travels to left}$$

Further, one could add a further phase angle to the argument as needed.