Lecture 31 - Diffusion

What's important:
- random walks
- diffusion

Demonstrations: 1 mL of food colouring placed in 2 L of still water; takes at least 45 minutes to diffuse, starting from the bottom of the flask

Diffusion equation

The concept of a random walk applies easily to the process of diffusion, where a particle moves randomly due to its collision with other particles. This might occur for a gas molecule banging against other molecules, or for a protein moving about in a cell. In either case, the motion is random.

Suppose that the diffusing particle makes one step of length b per unit time. Then the random walk tells us that the average end-to-end displacement of the walk is

$$\langle \mathbf{r}_{ee}^2 \rangle = b^2\, t \qquad \text{where } \langle .. \rangle \text{ indicates an average}$$

where t is the number of time steps. Now, the question is how big is b? For a gas molecule travelling fast in a dilute environment, b might be very large, but for a protein moving in a crowded cell, b is rather very small. We recognize this variation in b by writing the displacement as

$$\langle \mathbf{r}_{ee}^2 \rangle = 6D\, t \qquad \text{(diffusion in three dimensions)}$$

where D is the diffusion constant.

The factor of 6 is dimension-dependent. If an object diffuses in one dimension only (for example, a molecule moves randomly along a track) then

$$\langle \mathbf{r}_{ee}^2 \rangle = 2D\, t \qquad \text{(diffusion in one dimension)}$$

and if it is confined to a plane, such as a protein moving in the lipid bilayer of the cell's plasma membrane

$$\langle \mathbf{r}_{ee}^2 \rangle = 4D\, t \qquad \text{(diffusion in two dimensions)}$$

In any of these cases, D has units of $[length]^2 / [time]$.

For most fluids, D is in the range 10^{-14} to 10^{-10} m^2/s, depending on the size of the molecule. For the ATP molecule, which is the energy currency of the cell, $D \sim 3 \times 10^{-10}$ m^2/s.

Example How long does it take for a randomly moving protein to travel the distance of a cell diameter, say 10 μm, if its diffusion constant is 10^{-12} m^2/s?

Solving

$$t = (\mathbf{r}_{ee}^{\ 2})_{av} / 6D,$$

and

$$t = (10^{-5})^2 / 6 \cdot 10^{-12} = 16 \text{ seconds}$$

So it takes a protein less than a minute to diffuse across a cell at this diffusion constant; it would take much longer if the medium were more viscous and $D \sim 10^{-14}$ m^2/s.

Einstein equation

The diffusion constant can be determined analytically for a few specific situations. One case is the random motion of a sphere of radius R moving in a fluid of viscosity η, which Einstein solved using Stokes' Law: $F = 6\pi\eta R\, v$. We saw this formula for viscous drag back at the beginning of the course. The so-called Einstein relation reads:

$$D = k_B T / 6\pi\eta R$$

where k_B is Boltzmann's constant, having the numerical value $k_B = 1.38 \times 10^{-23}$ J/K. We'll see k_B in the kinetic theory of gases in Lec. 33. At room temperature (20 ^oC) where $T = 293$ K, the combination

$$k_B T = 293 \cdot 1.38\text{x}10^{-23} = 4\text{x}10^{-21} \text{ J}.$$

Now, $k_B T$ is close to the mean kinetic energy of a molecule, so the Einstein equation tells us that:

*the higher the temperature, the more kinetic energy an object has, the faster it diffuses.

*the larger an object is, or the more viscous its environment, the slower it diffuses.

Example A biological cell contains internal compartments with radii in the range 0.3 to 0.5 μm. Estimate their diffusion constant.

Solution. Suppose a cellular object like a vesicle has a radius of 0.3 μm and moves in a medium with viscosity $\eta = 2\text{x}10^{-3}$ kg / m•s. At room temperature, the Einstein relation predicts

$$D = 4\text{x}10^{-21} / (6\pi \cdot 2\text{x}10^{-3} \cdot 3\text{x}10^{-7}) = 4\text{x}10^{-13} \text{ m}^2/\text{s}.$$