Lecture 9 – Drag forces in cells

What's important:
- damped horizontal motion
- motion of a bacterium
- motion of a vesicle

Horizontal damped motion (linear regime)

Let's now solve the motion of an object subject only to linear drag in the horizontal direction – that is, omitting gravity. The initial speed of the object is v_o and it obeys Newton's law with linear drag

$$F_{drag} = ma \quad \text{--->} \quad -c_1 v = m\,(dv/dt) \quad \text{--->} \quad dv/dt = -(c_1/m)\,v \qquad (1)$$

where the - sign indicates that the force is in the opposite direction to the velocity. Given that we are now three weeks into this course, we use derivatives freely. This equation relates a velocity to its rate of change: it's solution is not a specific number like $v = 5$ m/s. Rather, it's solution gives the *form* of the function $v(t)$. It's easy to see that the solution is exponential in form, because

$$d\,e^x / dx = e^x.$$

That is, the derivative of an exponential is an exponential, satisfying Eq. (1). One still has to take care of the constants, and we can show using the chain rule that

$$v(t) = v_o \exp(-c_1 t / m). \qquad (2)$$

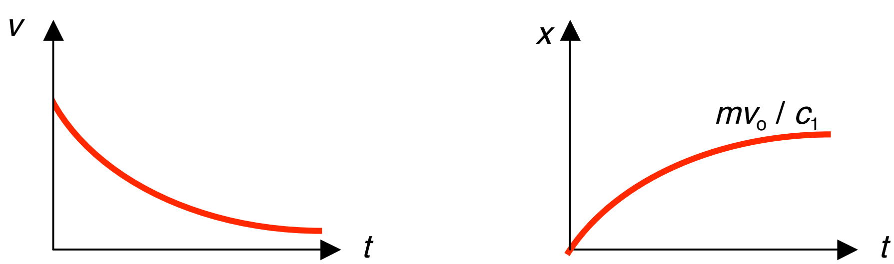
Clearly, this has the correct form at $t = 0$, namely

$$v(0) = v_o.$$

In general, we find

$$\begin{aligned} d/dt\,[v_o \exp(-c_1 t / m)] &= v_o\, d/dt\,[\exp(-c_1 t / m)] \\ &= v_o\, d[-c_1 t/m]/dt \cdot de^x/dx \qquad \text{where } x = -c_1 t / m \\ &= v_o\,(-c_1/m) \cdot e^x \\ &= (-c_1/m)\, v_o \exp(-c_1 t / m) \\ &= -c_1/m\; v(t) \end{aligned}$$

This last line is just Eq. (1) with a few rearrangements.



The characteristic time scale for the velocity to decay is m / c_1. Even though the velocity goes to zero only in the limit of infinite time, the object reaches a maximum position of mv_o/c_1, also at infinite time. The maximum distance can be found by integrating the *v vs. t* graph (need second term of calculus), yielding:

$$\Delta x = (mv_o / c_1) \cdot [\ 1 - \exp(-c_1 t / m)\ ]. \qquad \text{linear drag, no } g \qquad (3)$$

Example: drag force on a bacterium in water

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Many cells are capable of "swimming" in a fluid. Bacteria, for example, use long flagella or an array of short cilia as a basis for their movement. As described in PHYS 101 Supplement #2, flagella can rotate at tens of revolutions per second (like a car engine) and act like propellers to provide thrust to the cell. The bacterium *E. coli* is propelled by several flagella and can swim at 20 microns/second = 2×10^{-5} m/s.

Let's calculate the drag force on an idealized spherical bacterium swimming in water. For ease of calculation, we assume:
- the bacterium is a sphere of radius R = 1 micron
- the fluid medium is water with $\eta = 10^{-3}$ kg / m•s.
- the density of the cell is that of water $\rho = 1.0 \times 10^3$ kg/m^3.
- the speed of the bacterium is $v = 2 \times 10^{-5}$ m/s.

As before, we need to know the mass of the cell, and the prefactor c_1

$$\text{mass of cell} = \rho \cdot 4\pi R^3 /3 = 10^3 \cdot 4\pi\ (1 \times 10^{-6})^3 /3 = 4.2 \times 10^{-15} \text{ kg}.$$

$$c_1 = 6\pi\eta R = 6\pi \cdot 10^{-3} \cdot 1\times10^{-6} = 6\pi \cdot 10^{-9} = 1.9 \times 10^{-8} \text{ kg/s}.$$

We're now set to determine the maximum distance

$$x = mv_o / c_1 = 4.2\times10^{-15} \cdot 2\times10^{-5} / 1.9\times10^{-8} = 4.4 \times 10^{-12} \text{ m} = 0.04 \text{ Å}.$$

In other words, the cell comes to rest very fast because its motion is dominated by drag, not inertia. Now, there are several other quantities we can calculate for this situation:

What's the drag force on the cell? This is easy from our Stoke's law expression:

$F_{\mathrm{drag}} = c_1 v = 1.9\mathrm{x}10^{-8} \cdot 2\mathrm{x}10^{-5} = 3.8 \times 10^{-13}$ N = 0.4 pN. (pN = 10^{-12} N).

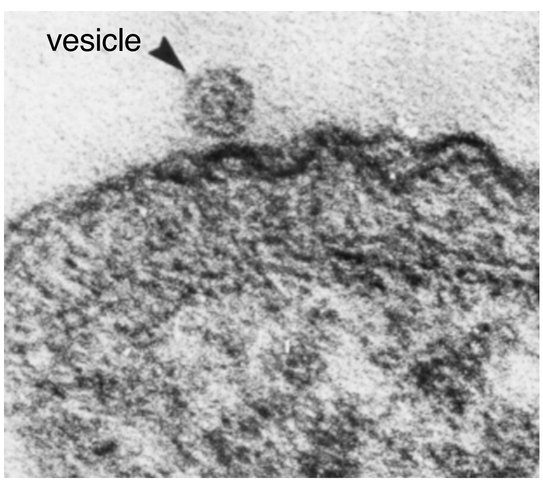
[*biological aside: the energy source for this type of motion probably does not involve direct hydrolysis of ATP, in contrast to the following example*]

Example: drag force on a vesicle in a cell

Transport of materials in our cells (but not bacteria) occurs in a directed fashion, like trucks on a highway. An extreme example is a nerve cell:

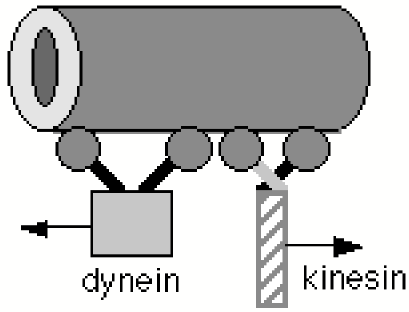
soma (main cell body) —— axon —— branched synapse

Neuroproteins and neurotransmitters (molecules) are produced at the soma, but consumed at the synaptic cleft, where they carry a signal from one nerve to another. Diffusion of chemicals along the axon (which may be a meter long!) would take an incredible amount of time, so nature packages the chemicals in small vesicles, which are transported along microtubules by molecular motors. Even with this mechanism, it may take several days for a neurotransmitter to be carried in a one-meter long neuron, from the soma in the brain to a synapse at a finger.



Typical vesicle radius would be 100 nm, with a lot of variation.

Two principle types of molecular motors are available to drag a vesicle along a microtubule. As seen from the diagram



The proteins *dynein* and *kinesin* have two attachment points to the microtubule, which in some ways are reminiscent of legs. The motion of the molecule involves the repeated side-stepping of these legs, more like a shuffle than a walk. For the dynein motor, the right leg would release its foothold on the microtubule, then reattach itself adjacent to the left leg. Then, the left leg would release, and take a step further along the filament before reattaching itself again. This side-step process then moves the motor along the microtubule. In the diagram, the vesicle would be attached to the "bottom" end of the motor molecule.

Example: What is the drag force that a molecular motor must overcome to transport a vesicle?

Assume:

R = 100 nm = 1 x 10^{-7} m

$\eta = 10^{-1}$ kg / m•s (say one hundred times more viscous than water)

v = 0.5 μm/s = 5 x 10^{-7} m/s

Thus

$c_1 = 6\pi\eta R = 6\pi \cdot 10^{-1} \cdot 1 \times 10^{-7} = 6\pi \cdot 10^{-8} = 1.9 \times 10^{-7}$ kg/s.

If $v = 5 \times 10^{-7}$ m/s

$F_{drag} = c_1 v = 1.9 \times 10^{-7} \cdot 5 \times 10^{-7} = 9 \times 10^{-14}$ N = 0.09 pN. (pN = 10^{-12} N)

What power is needed to propel the cell? Knowing that the power is the rate of change of energy, one finds from kinematics that power must also equal Fv. Here,

$P = F_{drag} v = 9.4 \times 10^{-14} \cdot 5 \times 10^{-7} \sim 5 \times 10^{-20}$ watts

Is this a lot of power or a little, by cellular standards? The cell's energy currency is a molecule called ATP (for adenosine triphophate), and which releases energy during hydrolysis to ADP (for adenosine diphophate), losing a phosphate in the process. The energy released in this reaction, which is strictly speaking the free energy released, is about 8 x 10^{-20} J. Thus, about one ATP molecule is needed per second to overcome the drag force in this hypothetical situation, assuming that all of this energy is available to the motor.

Example: The viscous drag force exerted by a stationary fluid on a spherical object of radius R is

$F = 6\pi\eta Rv$ at low speed (Stoke's law)

$F = (\rho/2)AC_D v^2$ at higher speeds,

Apply this to a spherical cell one micron in radius, moving in water with $\eta = 10^{-3}$ kg/m•s and $\rho = 10^3$ kg/m^3. Take the cell to have the same density as water, and let its drag coefficient be 0.5.

(a) Plot the two forms of the drag force as a function of cell speed up to 100 μm/s.

(b) Find the speed at which the linear and quadratic drag terms are the same.

Ans. (b) 24 m/s.

Example: Show that the terminal speed of a falling spherical object is given by

$v_{\text{term}} = [(mg/c_2) + (c_1/2c_2)^2\,]^{1/2} - (c_1/2c_2)$

when both the linear and quadratic terms in the drag force are taken into account.

Example: Consider three different power-law forms of the drag force with magnitudes:

$F_{1/2} = a\, v^{1/2}$ (square root)

$F_1 = b\, v^1$ (linear)

$F_{3/2} = c\, v^{3/2}$ (3/2 power).

Travelling horizontally from an initial speed v_o, an object subjected to one of these drag forces would come to rest at

$x_{\max} = (2m/3a)v_o^{3/2}$ (square root)

$x_{\max} = mv_o / b$ (linear)

$x_{\max} = 2m\, v_o^{1/2}/c$ (3/2 power).

(a) Determine the coefficents a, b and c for a cell of mass 10^{-14} kg whose drag force is measured to be 5 pN when travelling at 10 microns/second.

(b) Find the maximum displacement that the cell could reach for each force if v_o = 1 μm/s.

Ans. (b) 4.2 x 10^{-5} Å, 2 x 10^{-4} Å, 1.3 x 10^{-3}Å.

Example (*requires calculus*): (a) The linear drag force is parametrized by $F = c_1 v$. We stated in class that the position of an object obeying this force is described by

$x(t) = (mv_o/c_1) \bullet [\ 1 - \exp(-c_1 t/m)\ ]$.

Establish that the velocity corresponding to this $x(t)$ is

$v(t) = v_o \exp(-t/t_{\text{visc}})$,

with a characteristic time $t_{\text{visc}} = m/c_1$.

(b) The quadratic drag force is parametrized by $F = c_2 v^2$, resulting in

$x(t) = (v_o/k) \bullet \ln(1 + kt)$,

where

$k = c_2 v_o/m$.

Establish that the corresponding velocity is

$v(t) = v_o / (\ 1 + c_2 v_o t/m)$.

Does x reach a limiting value for quadratic drag?