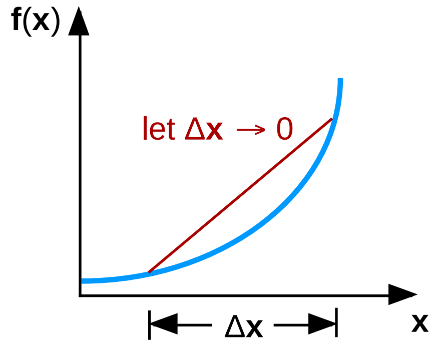
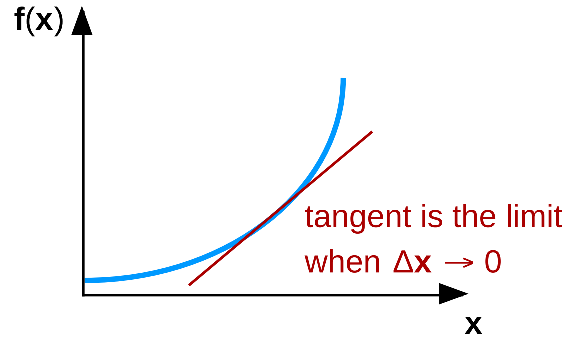
*Demonstrations:* none
*Text*: Fishbane Appendix IV

*What's important:*
•derivative of a polynomial function
•integral of a polynomial function

## Slopes

The slope of the tangent to a curve can be found from the slope of the chord bounding the point of interest, in the limit when the length of the chord vanishes:




Finding the slope to the tangent is called "taking the derivative". Suppose that we have a polynomial function $\mathbf{f}(\mathbf{x}) = \mathbf{x}^n$. Then the slope of the tangent at $\mathbf{x}$ is:

$$\text{slope of tangent} = \frac{\mathbf{f}(\mathbf{x}+\Delta\mathbf{x}) - \mathbf{f}(\mathbf{x})}{\Delta\mathbf{x}} \quad \text{as } \Delta\mathbf{x} \rightarrow 0$$

$$= \frac{(\mathbf{x}+\Delta\mathbf{x})^n - \mathbf{x}^n}{\Delta\mathbf{x}}$$

$$= \frac{\mathbf{x}^n(1 + \Delta\mathbf{x}/\mathbf{x})^n - \mathbf{x}^n}{\Delta\mathbf{x}}$$

$$\frac{\mathbf{x}^n(1 + n\bullet\Delta\mathbf{x}/\mathbf{x}) - \mathbf{x}^n}{\Delta\mathbf{x}} \quad \text{since } \Delta\mathbf{x}/\mathbf{x} \text{ is small}$$

$$= \frac{\mathbf{x}^n + n\bullet\mathbf{x}^n\bullet\Delta\mathbf{x}/\mathbf{x} - \mathbf{x}^n}{\Delta\mathbf{x}}$$

$$= n\bullet\mathbf{x}^{n-1}$$

Thus, the derivative, or the slope of the tangent to the curve, is $n\mathbf{x}^{n-1}$. We denote the derivative as $d\mathbf{f}(\mathbf{x}) / d\mathbf{x}$, where $d\mathbf{x}$ is to be read as one symbol, not the product of a variable **d** with a variable **x**. Thus,

$$\frac{d\mathbf{f}(\mathbf{x})}{d\mathbf{x}} = n \bullet \mathbf{x}^{n-1}$$

If the function is not just $\mathbf{f}(\mathbf{x}) = \mathbf{x}^n$, but has an overall multiplicative constant **a**, then a repeat of the above calculation shows that

$$\frac{d(\mathbf{a}\mathbf{x}^n)}{d\mathbf{x}} = \mathbf{a} n \bullet \mathbf{x}^{n-1}$$

The derivatives of some other functions that we use in this course include:

$d \sin\theta / d\theta = \cos\theta$

$d \cos\theta / d\theta = -\sin\theta$

$d \exp(\mathbf{x}) / d\mathbf{x} = \exp(\mathbf{x})$.

Other aspects of taking derivatives, such as the chain rule, are covered in standard calculus texts.

**Example** Suppose the distance that a car moves as a function of time is given by $\mathbf{L}(\mathbf{t}) = 20\mathbf{t}^2$, where **t** is in [sec] and **L** is in [meters]. Find $\mathbf{v}(\mathbf{t})$ and $\mathbf{a}(\mathbf{t})$.

First, note that there are implicit units attached to the "20": it is 20 [m/s$^2$], so that 20 [m/s$^2$]$\bullet\mathbf{t}^2$ gives an answer in meters.

Then,

$$\mathbf{v}(\mathbf{t}) = d\mathbf{x}(\mathbf{t}) / d\mathbf{t} = d(20\,\mathbf{t}^2) / d\mathbf{t} = 20 \bullet 2 \bullet \mathbf{t}^{2-1} = 40\mathbf{t} \quad \text{[m/s]}$$

and

$$\mathbf{a}(\mathbf{t}) = d\mathbf{v}(\mathbf{t}) / d\mathbf{t} = d(40\,\mathbf{t}) / d\mathbf{t} = 40 \bullet 1 \bullet \mathbf{t}^{1-1} = 40 \quad \text{[m/s}^2\text{]}$$

Values for **x** and **v** at any time **t** can be found by substituting the desired value of **t**.

### Areas and integration

In our discussion of kinematics in one dimension, we said that finding the slope to a curve, and taking the area under a curve, were in some sense inverse processes. That is, we said

$$\mathbf{x} \quad \text{---slope--->} \quad \mathbf{v} \qquad\qquad \mathbf{v} \quad \text{---area--->} \quad \mathbf{x}$$

or, in derivative language

$$\mathbf{x} \quad \text{---slope--->} \quad d\mathbf{x} / d\mathbf{t} \qquad\qquad d\mathbf{x} / d\mathbf{t} \quad \text{---area--->} \quad \mathbf{x}$$

or, changing notation to functions of a general variable **x** (rather than the kinematics variable of time **t**)

$$\mathbf{f}(\mathbf{x}) \quad \text{---slope--->} \quad d\mathbf{f}(\mathbf{x}) / d\mathbf{x} \qquad\qquad d\mathbf{f}(\mathbf{x}) / d\mathbf{x} \quad \text{---area--->} \quad \mathbf{f}(\mathbf{x})$$

We want to find an analytical expression for the area under a polynomial function. According to our ideas about slopes and areas, if d**f**(**x**) / d**x** is the slope of the function **f**(**x**) (where we use the word "slope" to mean "slope of the tangent to the curve") then the area under the curve d**f**(**x**) / d**x** must be **f**(**x**). That is,

$$[\text{area of } d\mathbf{f}(\mathbf{x}) / d\mathbf{x}] = \mathbf{f}(\mathbf{x})$$

or, applying this to a polynomial

$$[\text{area of } n \bullet \mathbf{x}^{n-1}] = \mathbf{x}^{n}$$

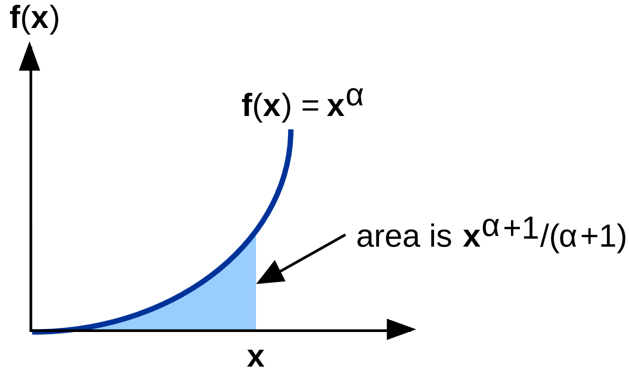
Changing variables so that $\alpha = n - 1$, or $n = \alpha + 1$, then

$$[\text{area of } (\alpha + 1) \bullet \mathbf{x}^{\alpha}] = \mathbf{x}^{\alpha+1}$$

or

$$[\text{area of } \mathbf{x}^{\alpha}] = \mathbf{x}^{\alpha+1} / (\alpha + 1)$$

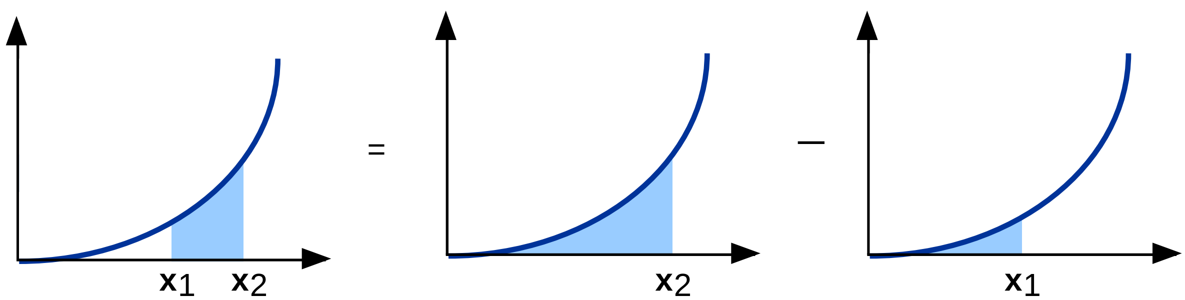
This expression is valid if by the words "area under the curve" we mean area in the range 0 to **x**:




However, in general we are interested in the area between specified values of $\mathbf{x}_1$ and $\mathbf{x}_2$. This can be found by subtraction:

[area of $\mathbf{f}(\mathbf{x})$ between $\mathbf{x}_1$ and $\mathbf{x}_2$] =
[area of $\mathbf{f}(\mathbf{x})$ between 0 and $\mathbf{x}_2$] - [area of $\mathbf{f}(\mathbf{x})$ between 0 and $\mathbf{x}_1$]

Graphically, this can be seen as




Thus,

$$[\text{area of } \mathbf{f}(\mathbf{x}) \text{ between } \mathbf{x}_1 \text{ and } \mathbf{x}_2] = (\mathbf{x}_2^{\alpha+1} - \mathbf{x}_1^{\alpha+1}) / (\alpha + 1)$$

We finish off with some notation. The area under the curve is referred to as an integral. We can think of evaluating the area by breaking up the range of $\mathbf{x}$ from $\mathbf{x}_1$ to $\mathbf{x}_2$ into many small segments *i*, each having some specified width $d\mathbf{x}_i$ centred at $\mathbf{x}_i$. The approximate area under the curve at each $\mathbf{x}_i$ is $\mathbf{f}(\mathbf{x}_i) \bullet d\mathbf{x}_i$. Finally, all the small area elements can be summed together to yield the total area:

$$[\text{area of } \mathbf{f}(\mathbf{x}) \text{ between } \mathbf{x}_1 \text{ and } \mathbf{x}_2] \sim \Sigma_i \, \mathbf{f}(\mathbf{x}_i) \cdot d\mathbf{x}_i$$

In the limit when the segments become very small, this approximation becomes exact, and the summation sign $\Sigma_i$ is replaced by an integral sign $\int$.

$$[\text{area of } \mathbf{f}(\mathbf{x}) \text{ between } \mathbf{x}_1 \text{ and } \mathbf{x}_2] = \int_{\mathbf{x}_1}^{\mathbf{x}_2} \mathbf{f}(\mathbf{x}) \, d\mathbf{x}$$