Demonstrations:
•collisions on an air track
Text: Fishbane 8-1, 8-2, 8-3, 8-4, 8-5, 8-6
Problems: 12, 28, 38, 51, 58 from Ch. 8

What's important:
•conservation of momentum
•centre of mass motion
•collisions in one dimension

Momentum, mass and position of a many-particle system

The kinematics that we have introduced so far is that of a single particle, using primary variables **x**, **v**, and **a** to describe the position and motion of a particle. From these quantities, we obtain the kinetic energy K = $mv^2/2$ and momentum **p** = m**v**. What we now want to do is describe a system containing many particles, say stars in a galaxy or molecules in a gas. We need to find variables like **x**, **v**, and **a** to characterize the system as a whole.

Let's start by examining the total momentum of the system $\mathbf{P}_{tot}$,

$$\vec{P}_{tot} = \sum_i \vec{p}_i = \sum_i m_i \vec{v}_i = \sum_i m_i \frac{d\vec{r}_i}{dt} = \frac{d}{dt} \underbrace{\sum_i m_i \vec{r}_i}_{\text{This is a vector}}$$

Note that this is a vector equation, $\mathbf{P}_{tot}$ is not the scalar sum of the magnitudes of the individual momenta. Multiplying and dividing the right-hand-side by the total mass M_{tot} gives

$$\vec{P}_{tot} = M_{tot} \frac{d}{dt} \frac{1}{M_{tot}} \left(\sum_{i=1}^{N} m_i \vec{r}_i \right) .$$

This expression resembles **P** = **mv** = **m** d**r**/d**t**, where the momentum is P_{tot}, and the mass is M_{tot}. Hence, the quantity

$$\frac{1}{M_{tot}} \left(\sum_{i=1}^{N} m_i \vec{r}_i \right)$$

represents the effective position of the many-particle system as a whole. We refer to this quantity as R_{cm}, the **centre-of-mass position**:

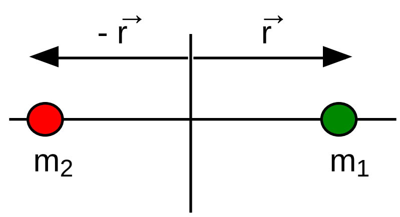
$$\vec{R}_{cm} = \frac{1}{M_{tot}} \left(\sum_{i=1}^{N} m_i \vec{r}_i \right)$$

What appears in our expression for $\mathbf{P}_{tot}$ is the derivative of R_{cm}, which is the centre-of-mass velocity:

$$\vec{V}_{cm} = \frac{d}{dt} \vec{R}_{cm} = \frac{d}{dt} \frac{1}{M_{tot}} \left(\sum_{i=1}^{N} m_i \vec{r}_i \right)$$

The centre-of-mass position is the weighted average of the positions of the individual components of the system, and describes the motion of the system as a whole.

Example: Consider two unequal mass objects, m_1 and m_2, at positions +r and -r:



$$\vec{R}_{cm} = \frac{1}{m_1 + m_2} (m_1 \vec{r} - m_2 \vec{r}) = \frac{m_1 - m_2}{m_1 + m_2} \vec{r}$$

Two special cases:

Suppose $m_1 = m_2$, then $\mathbf{R}_{cm} = 0$ (i.e., the cm sits at the coordinate origin).

Suppose $m_1 >> m_2$, then $m_1 + m_2 \sim m_1$ and $\vec{R}_{cm} \sim (m_1 / m_1)\, \vec{r} = \vec{r}$.

Motion of the cm

What determines the motion of the centre-of-mass? For a single particle, we have $\mathbf{F} = m\mathbf{a}$, which we can also write as

$$\mathbf{F} = d\mathbf{p} / dt.$$

What happens in a many-body system? The particles are each subject to individual forces $\mathbf{F}_i$, such that the total force on the system is

$$F_{net} = \sum_i \vec{F}_i$$

Then by Newton's Second Law,

$$\vec{F}_{net} = \sum_i \vec{F}_i = \sum_i \frac{d}{dt}\vec{p}_i = \frac{d}{dt}\sum_i \vec{p}_i = \frac{d}{dt}\vec{P}_{Tot}$$

or

$$\vec{F}_{net} = \frac{d}{dt}\vec{P}_{Tot}$$

But this expression is just like **F** = d**p**/dt, and it says that the total momentum obeys a dynamical equation of the same form as Newton's second law. In other words, the total momentum is constant unless the system is acted upon by a net external force.

We know $P_{Tot} \equiv M_{Tot}(dR_{cm} / dt)$ from above work, so we find that

$$\vec{F}_{net} = \frac{d}{dt}\left(M_{Tot}\frac{d}{dt}\vec{R}_{cm}\right)$$

$$= \frac{d}{dt}\left(M_{Tot}\vec{V}_{cm}\right) \qquad \vec{V}_{cm} = \text{cm velocity} \equiv \frac{d\vec{R}_{cm}}{dt}$$

$$= M_{Tot}\frac{d\vec{V}_{cm}}{dt} \qquad \vec{a}_{cm} = \frac{d\vec{V}_{cm}}{dt}$$

$$= M_{Tot}\,\vec{a}_{cm}$$

Thus, $\vec{R}_{cm}$, $\vec{V}_{cm}$, $\vec{a}_{cm}$ behave just like any kinematic set $\vec{\mathbf{r}}$, $\vec{\mathbf{v}}$, $\vec{\mathbf{a}}$, except that the dynamics is governed by $\vec{F}_{net}$. This is why we don't need to worry about the dynamics of quarks when we talk about the motion of a car.

Conservation of Momentum

In the previous lecture, we dealt with a fundamental conservation law of Nature: conservation of energy. There is another equally important conservation law - conservation of momentum. Conservation of momentum has very deep roots in our understanding of space and arises from the hypothesis that the laws of physics are the same everywhere in the universe. Conservation of momentum is a vector equation, which says that for a system of **N** particles, the total momentum

$$\vec{P}_{Tot} = \sum_{i=1}^{N} \vec{p}_i$$

does not change with time

$$\frac{d\,\vec{P}_{Tot}}{dt} = 0$$

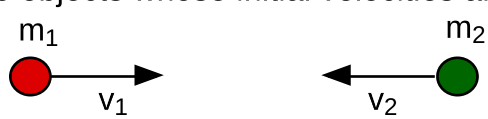
Consider the total momentum of two particles (**N** = 2 in the above equation), then

$$0 = \frac{d\,\vec{P}_{Tot}}{dt} = \frac{d}{dt}\left(\vec{p}_1 + \vec{p}_2\right) = \vec{F}_1 + \vec{F}_2$$

Thus, $\vec{F}_1 = -\vec{F}_2$, as expected from Newton's Third Law.

Collisions in One Dimension

We now wish to apply the conservation of energy and momentum to the interaction of objects. Consider two objects whose initial velocities and masses are known:



After the objects interact (or in this case, collide), we have

Can we determine v_1' and v_2'? We know that momentum is conserved, so

$$m_1v_1 + m_2\underline{\underline{v_2}} = m_1v_1' + m_2v_2'$$

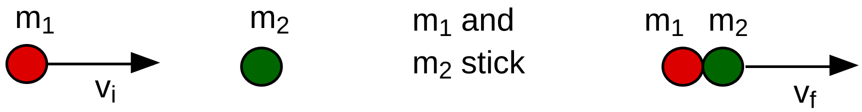
└ take sign into account

If the collision involves no dissipative forces, then we also have conservation of kinetic energy

$$\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 = \frac{1}{2}m_1(v_1')^2 + \frac{1}{2}m_2(v_2')^2$$

So, in this situation we have two equations and two unknowns, and we can solve for both v_1' and v_2' .

Consider a slightly different situation:



We can solve this by conservation of momentum alone:

$$m_1 v_i + 0 = (m_1 + m_2)\, v_f \qquad \text{or} \qquad v_f = \frac{m_1}{m_1 + m_2}\, v_i$$

But conservation of kinetic energy also gives an equation relating v_i and v_f. Is this equation consistent with the results from conservation of momentum? To answer this question, we evaluate the kinetic energy before and after the collision as determined by the conservation of momentum equation:

$$K_i = \frac{1}{2} m_1 v_i^2 \qquad K_f = \frac{1}{2} (m_1 + m_2)\, v_f^2$$

$$= \frac{1}{2} (m_1 + m_2) \left(\frac{m_1}{m_1 + m_2}\right)^2 v_i^2$$

$$= \frac{1}{2} \left(\frac{m_1}{m_1 + m_2}\right) m_1 v_i^2$$

$$= \frac{m_1}{m_1 + m_2} K_i$$

$K_f < K_i$ and kinetic energy is not conserved. The difference in kinetic energy between the initial and final states has gone into heat or sound or whatever.

Rules:
•first apply conservation of momentum (**vector**, results in 1 to 3 equations).
•then evaluate the kinetic energies (1 equation)

If the collision is **elastic**, then kinetic energy is conserved. If kinetic energy is not conserved, then the collision is **inelastic** and $K_f < K_i$. Thus, kinetic energy may not provide a constraint on the values of the momenta after the collision. Of course, even if kinetic energy is not conserved, the **total energy**, including heat *etc.*, must be conserved.

2-Body Collisions in Three Dimensions

This situation cannot be solved by conservation laws alone:
- •final state has 6 unknowns (3 components of velocity for two particles)
- •conservation laws provide 3+1 equations

Need information about the interaction between particles, for example, the potential energy function.