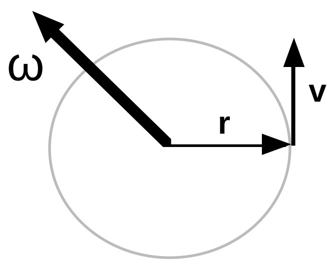
Demonstrations: none
Text: Fishbane 9-4, 9-5, 9-6, 10-1, 10-2, 10-3
Problems: 38, 42, 43 from Ch. 9; 18, 20 from Ch. 10

What's important:
•angular momentum and torque
•summary of angular kinematics and dynamics
•links between angular and linear equations

Vectors for $\vec{\omega}$, $\vec{\alpha}$:

We have said for circular motion that $v = \omega r$. In fact, in vector form $\vec{\mathbf{v}} = \vec{\omega} \times \vec{\mathbf{r}}$.



(use right hand rule on $\vec{\omega} \times \vec{\mathbf{r}}$ to find how $\vec{\mathbf{v}}$ points.)

NOTE: the cross product cannot be inverted to read $\omega = \mathbf{v} / \mathbf{r}$.

The link between α and $\mathbf{a}_t$ is found by expressing the total acceleration vector in terms of its tangential and centripetal components:

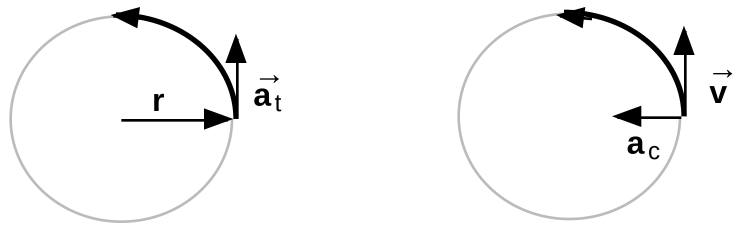
$$\vec{\mathbf{a}}_t + \vec{\mathbf{a}}_c = \frac{d\vec{\mathbf{v}}}{dt} = \frac{d\vec{\omega}}{dt} \times \vec{\mathbf{r}} + \vec{\omega} \times \frac{d\vec{\mathbf{r}}}{dt}$$

$$\vec{\alpha} \times \vec{\mathbf{r}} \equiv \vec{\mathbf{a}}_t \qquad\qquad \vec{\omega} \times \vec{\mathbf{v}} \qquad \vec{\mathbf{a}}_c = \vec{\omega} \times \vec{\mathbf{v}}$$

must be $\vec{\mathbf{a}}_t$ since it is perpendicular to $\vec{\mathbf{r}}$.

must be $\vec{\mathbf{a}}_c$ since it is perpendicular to $\vec{\mathbf{v}}$.

To confirm the orientation of the tangential and centripetal acceleration, recall

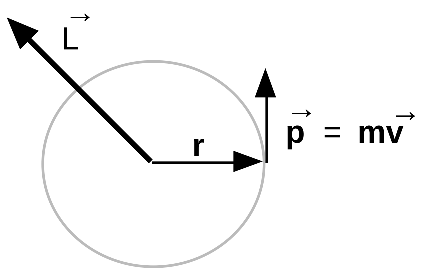


Thus, we have

$$\vec{\alpha} \times \vec{\mathbf{r}} \equiv \vec{\mathbf{a}}_t$$

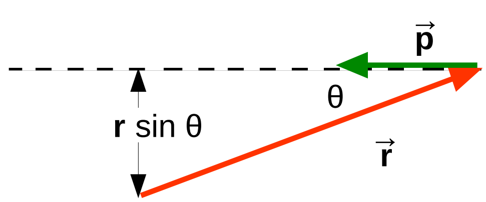
Angular Momentum

There are angular analogues of momentum and force. First, angular momentum $\vec{L}$ is $\vec{\mathbf{r}} \times \vec{\mathbf{p}}$.



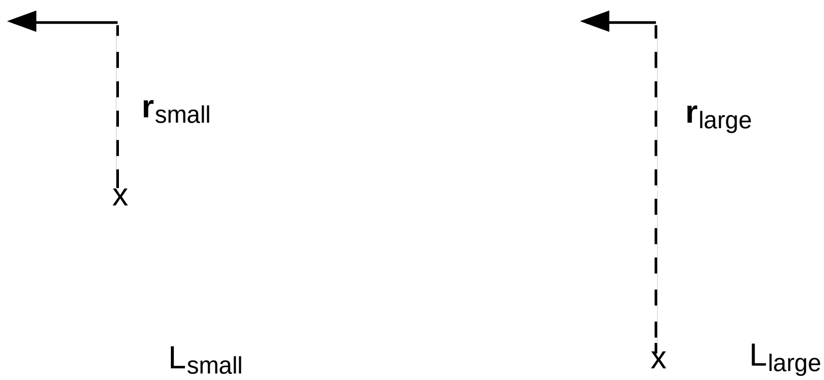
i) Note that we can also write this as

$$L = \mathbf{r}\,\mathbf{p}\sin\theta = (\mathbf{r}\sin\theta)\mathbf{p} = \mathbf{r}\quad\mathbf{p}$$



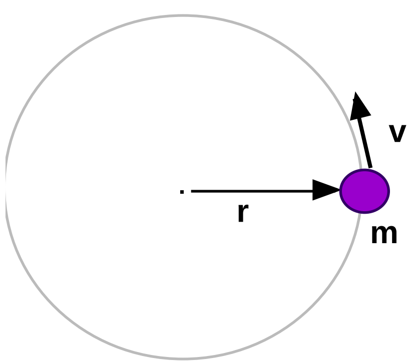
ii) Note, if $\vec{\mathbf{r}}$ is parallel to $\vec{\mathbf{p}}$, then the angular momentum vanishes since $\theta = 0$.

iii) Note: $\vec{L}$ is defined with respect to some point in space.



Moment of inertia

The angular momentum **L** is related to the angular velocity ω through the moment of inertia. Consider the motion of a single mass **m** around a point:



Now,

$$\mathbf{L} = \mathbf{rp}$$
$$= \mathbf{r}\ \mathbf{mv}$$
$$= \mathbf{r}\ \mathbf{m}\ \omega\mathbf{r}$$
$$= \underline{\underline{(\mathbf{mr}^2)}}\ \omega$$

This is called the moment of inertia *I* or rotational inertia.

$$\vec{\mathbf{L}} = I\,\vec{\omega} \text{ just like } \vec{\mathbf{p}} = \mathbf{m}\vec{\mathbf{v}}$$

Torque

In linear kinematics, force is the rate of change of momentum: **F** = *m* d**p** / d*t*. In angular motion, **torque** is the rate of change of **angular momentum**:

$$\vec{\tau} = \frac{d\vec{L}}{dt} = \frac{d\, I\vec{\omega}}{dt} = I\frac{d\vec{\omega}}{dt} = I\vec{\alpha}$$

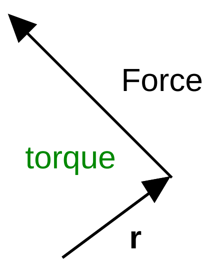
To link $\vec{\tau}$ to $\vec{F}$, evaluate the derivative in full:

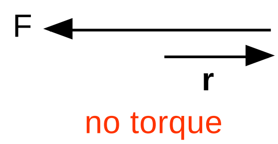
$$\vec{\tau} = \frac{d\vec{L}}{dt} = \frac{d\; \vec{\mathbf{r}} \times \vec{\mathbf{p}}}{dt} = \underbrace{\left(\frac{d\vec{\mathbf{r}}}{dt}\right)}_{\vec{\mathbf{v}}} \times \underbrace{\vec{\mathbf{p}}}_{m\vec{\mathbf{v}}} + \vec{\mathbf{r}} \times \frac{d\vec{\mathbf{p}}}{dt} = 0 + \vec{\mathbf{r}} \times \vec{F}$$

v x **v** is a cross product of a vector with itself and hence vanishes.

$$\vec{\tau} = \vec{\mathbf{r}} \times \vec{F} \qquad \text{or} \qquad \tau = \mathbf{r}\, F$$

The maximum torque occurs when **r** is perpendicular to **F**. If **r** is parallel to **F**, then there is no torque (the force still acts, but there does not change the angular momentum):

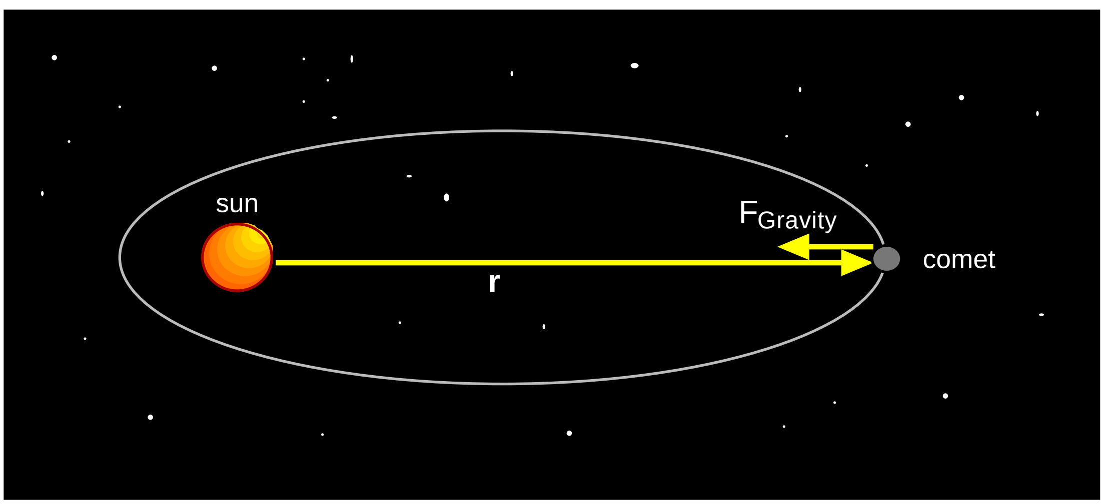




Summary

Linear	Links	Angular
$\vec{\mathbf{v}} = \dfrac{d\vec{\mathbf{x}}}{dt}$	$\vec{\mathbf{v}} = \vec{\omega} \times \vec{\mathbf{r}}$	$\vec{\omega} = \dfrac{d\theta}{dt}$
$\vec{\mathbf{a}} = \dfrac{d\vec{\mathbf{v}}}{dt}$	$\vec{\mathbf{a}}_t = \vec{\alpha} \times \vec{\mathbf{r}}$	$\vec{\alpha} = \dfrac{d\vec{\omega}}{dt}$
$\vec{\mathbf{p}} = m\vec{\mathbf{v}}$	$\vec{L} = \vec{\mathbf{r}} \times \vec{\mathbf{p}}$	$\vec{L} = \underline{\underline{I}}\vec{\omega}$ moment of inertia
$\vec{F} = \dfrac{d\vec{\mathbf{p}}}{dt}$	$\vec{\tau} = \vec{\mathbf{r}} \times \vec{F}$	$\vec{\tau} = \dfrac{d\vec{L}}{dt}$

Example: Motion of a comet around the sun



$$\vec{\mathbf{r}} \times \vec{F} = 0, \quad \text{since they are parallel.}$$

$$\frac{d\vec{L}}{dt} = 0 \quad \text{or } L \text{ is a constant.}$$

Note that the linear momentum **p** changes in magnitude during the orbit, even though the angular momentum is constant.