*Demonstrations:*
•bike wheel, rotating stool, gyroscope
*Text*: Fishbane 9-4, 10-4, 10-7
*Problems*: 23, 27, 31, 47, 49 from Ch. 9

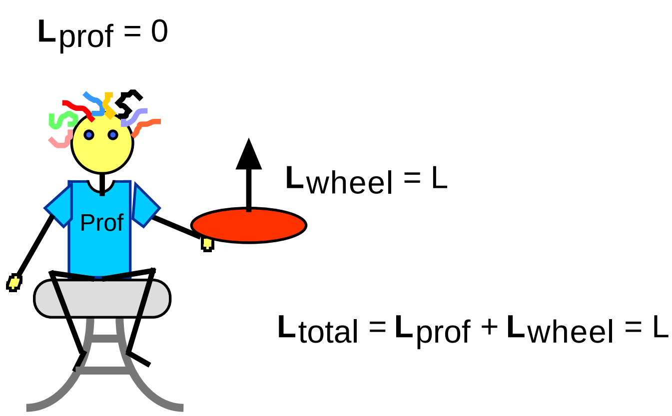
*What's important:*
•conservation of angular momentum
•moments of inertia
•parallel and perpendicular axis theorems

## Conservation of angular momentum

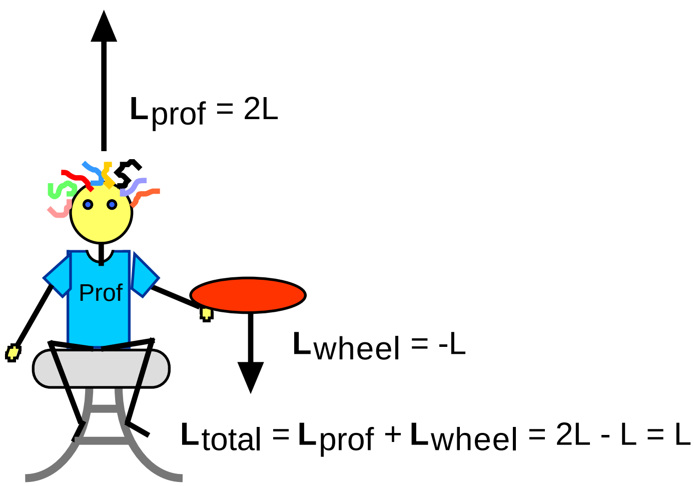
We saw previously that linear momentum **p** is conserved unless there is a net external force: the applied force is the rate of change of linear momentum. So too with angular momentum: **Angular momentum L is conserved unless there is a net external torque**.

As an example of conservation of angular momentum, consider the following demonstration. Prof sits on a stool which is free to rotate, and holds a rotating object in his hands: in the diagram, a bike wheel. The rotating object has an angular momentum, pointing in a particular direction. Prof changes the direction of the angular momentum of the object by changing its orientation. But the total angular momentum of the system is conserved, with the result that the angular momentum of the prof must change in an equal and opposite way to the change of the rotating object. As a result, the prof begins to rotate on the stool.

BEFORE



AFTER

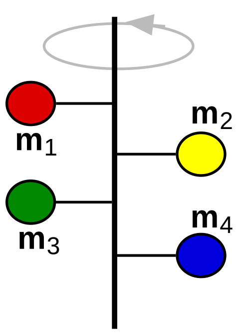



The total angular momentum after the change in orientation is the same as that before.

Now consider how conservation of angular momentum applies to a situation in which the moment of inertia changes. The moment of inertia **I** for a single mass **m** executing a circle of radius **r** about an axis is

$$\mathbf{I} = \mathbf{mr}^2.$$

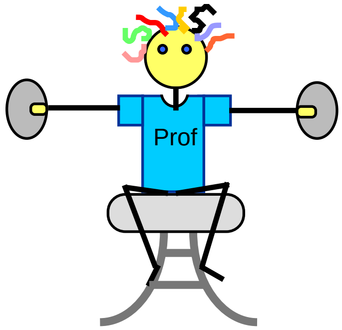
For a group of masses, all rotating with the same $\omega$:



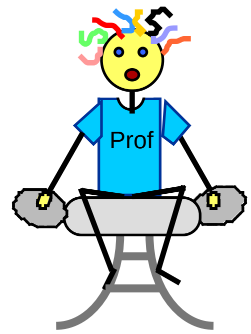

$$I = \mathbf{m}_1\mathbf{r}_1^2 + \mathbf{m}_2\mathbf{r}_2^2 + \mathbf{m}_3\mathbf{r}_3^2 + \mathbf{m}_4\mathbf{r}_4^2$$

$$= \sum_i \mathbf{m}_i \mathbf{r}_i^2$$

In the demo, a slowly rotating prof has a big moment of inertia by holding weights out at the end of his arms. Dropping his arms into the vertical position reduces his moment of inertia. Since angular momentum is conserved, then the prof's angular velocity must increase as his moment of inertia decreases:




big $I$

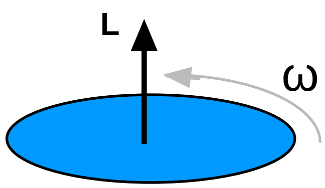



small $I$

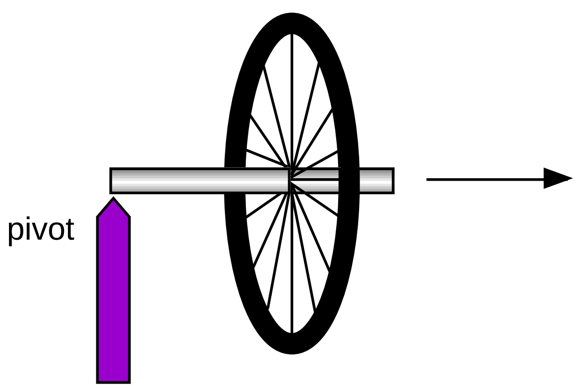
**L** is conserved. $\mathbf{L} = I_{big}\,\omega_{small} = I_{small}\,\omega_{big}$

**Demo:** gyroscope.

The gyroscope is a spinning top with angular momentum **L**.

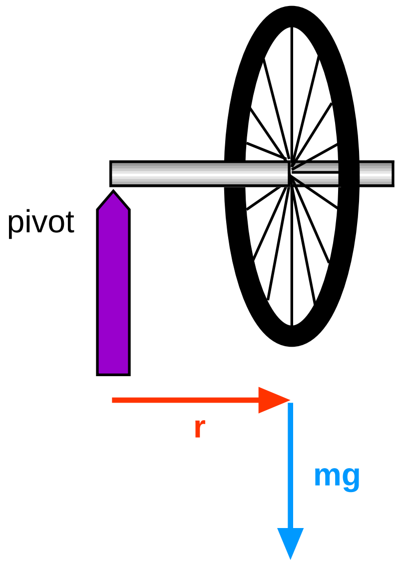



Consider what happens when we place the axle of a spinning bike wheel on a pivot:

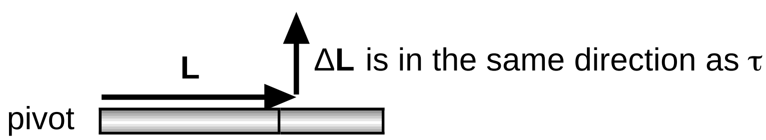



$\vec{L} = I\vec{\omega}$

The torque from gravity points into the plane of the diagram, because the force from gravity on the bike is a distance from the pivot point (torque is perpendicular to **r** and **mg** in the diagram):




Because the torque is perpendicular to L, it causes L to rotate, but not change direction.  Looking down on the bike wheel (omitted in the diagram for clarity):




### Moments of Inertia in Detail

The moment of inertia calculation for point - like masses is straight forward:

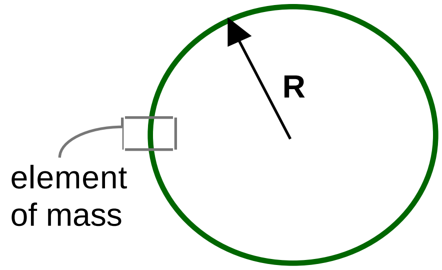
$$I = \sum \mathbf{m}_i \mathbf{r}_i^2$$

If the mass distribution is continuous, then the sum over masses becomes an integral:

$$\sum \mathbf{r}_i^2 \, \Delta\mathbf{m} \qquad \int \mathbf{r}^2 \, d\mathbf{m}$$

Several examples follow.

i ) Ring of radius **R**, mass **M**.




Break up the mass of the ring into **N** small segments each of mass

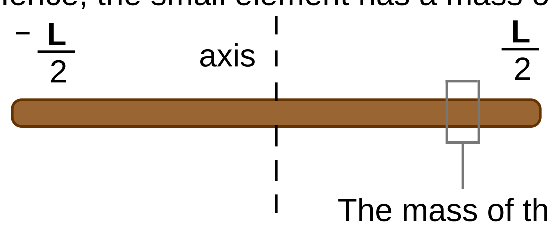
$$\mathbf{M} \qquad \mathbf{m} = \frac{\mathbf{M}}{\mathbf{N}}$$

Each mass element **m** is a distance **R** away from the axis of rotation.
Then **I** is just the sum over the small elements of mass

$$I = \sum_i \mathbf{m}_i \mathbf{R}^2 = \mathbf{R}^2 \sum_i \mathbf{m}_i = \mathbf{R}^2 \mathbf{M}$$

ii) Thin rod

We break up the rod into small elements of length d**x**. The mass per unit length along the rod is $\lambda = \mathbf{M}/\mathbf{L}$. Hence, the small element has a mass of d**m** = $\lambda$ d**x**.



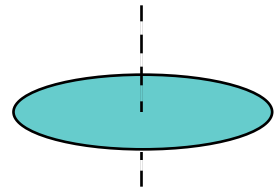

$$I = \int_{-\frac{L}{2}}^{\frac{L}{2}} \mathbf{x}^2\, d\mathbf{m} = \int_{-\frac{L}{2}}^{\frac{L}{2}} \mathbf{x}^2\, \lambda\, d\mathbf{x} = \lambda \left.\frac{\mathbf{x}^3}{3}\right|_{-\frac{L}{2}}^{\frac{L}{2}}$$

$$= \frac{\lambda}{3}\left[\frac{\mathbf{L}^3}{8} - \left(-\frac{\mathbf{L}^3}{8}\right)\right] = \frac{(\lambda\,\mathbf{L})\,\mathbf{L}^2}{12} = \frac{\mathbf{M}\,\mathbf{L}^2}{12}$$

iii )  Disk  (more calculus!)

$$I = \frac{1}{2} \mathbf{M\,R}^2$$

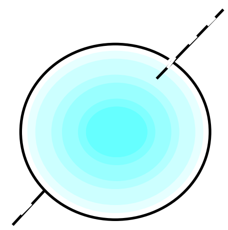
(axis perpendicular to plane)



iv )  Solid Sphere  (still calculus!)

$$I = \frac{2}{5} \mathbf{M\,R}^2$$

(axis through centre)

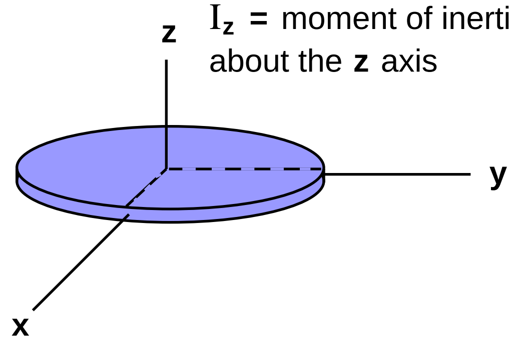


**Other Axes of Rotation**

All moments of inertia are defined with respect to a particular axis.  What happens if the axis of rotation is not one of the simple ones?  Two theorems help (proofs in most first year texts):

1.)  Perpendicular Axis Theorem for flat objects in **xy** plane

$$I_z = I_x + I_y$$

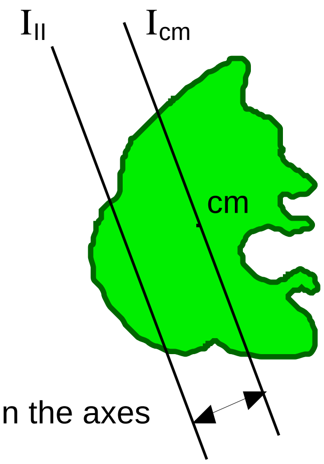



$I_z$ = moment of inertia about the **z** axis

2.) Parallel Axis Theorem

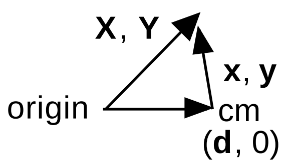
for two parallel axes, one of which goes through the centre - of - mass.

$$I_{II} = I_{cm} + \mathbf{M}\,\mathbf{d}^2$$




**d** = distance between the axes

Proof:



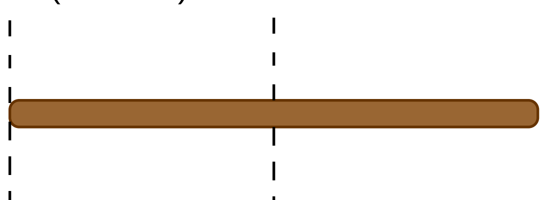

$$\mathbf{X}^2 + \mathbf{Y}^2 = (\mathbf{d} + \mathbf{x})^2 + \mathbf{y}^2 = \mathbf{d}^2 + 2\mathbf{dx} + \mathbf{x}^2 + \mathbf{y}^2$$

$$\sum \mathbf{m}(\mathbf{X}^2 + \mathbf{Y}^2) = \mathbf{Md}^2 + 2\mathbf{d}\sum \mathbf{mx} + \sum \mathbf{m}(\mathbf{x}^2 + \mathbf{y}^2)$$

$$I_{II} = \mathbf{Md}^2 + I_{cm}$$

$\sum \mathbf{mx} = 0$ for cm

**Example** (Thin rod)



$I_{II} = ?$

$I_{cm} = \frac{1}{12}\,\mathbf{ML}^2$

$$I_{II} = I_{cm} + \mathbf{Md}^2$$

$$= \frac{1}{12}\,\mathbf{ML}^2 + \mathbf{M}\left(\frac{\mathbf{L}}{2}\right)^2$$

$$= \left(\frac{1}{12} + \frac{1}{4}\right)\mathbf{ML}^2$$

$$I_{II} = \frac{1}{3}\,\mathbf{ML}^2 > I_{cm}$$

Note: This tells us that the axis through the **cm** has the smallest moments of inertia (in a given direction).