Demonstrations:
•Atwood's machine
•giant yo-yo on a table
•cylinders on an incline plane
Text: Fishbane 9-5, 10-3
Problems: 53, 60, 64 from Ch. 10

What's important:
•Atwood's machine, rolling without slipping

Dynamics with Rotation

Our equations for rotational dynamics read:

$$\vec{\mathbf{L}} = I\,\vec{\omega} \quad (= \vec{\mathbf{r}} \times \vec{\mathbf{p}})$$

$$\vec{\tau} = I\,\vec{\alpha} \quad (= \vec{\mathbf{r}} \times \vec{\mathbf{F}}) \qquad \vec{\tau} = \frac{d\vec{\mathbf{L}}}{d\mathbf{t}}$$

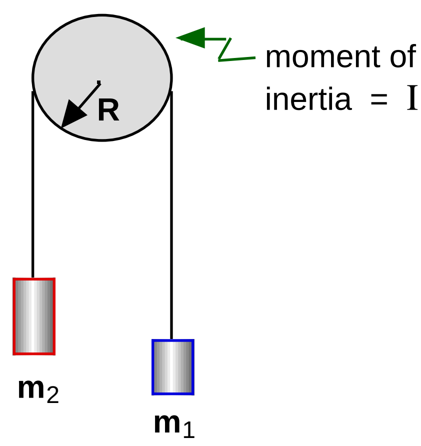
$$\text{K.E.} = \frac{1}{2} I \omega^2$$

Let's apply these equations to several examples of rotating systems.

<u>Example 1: Atwood's Machine</u> (used for measuring **g**)

A string with a mass at each end runs without slipping over a pulley. As the pulley accelerates, so do the masses.

Find the acceleration of the masses.



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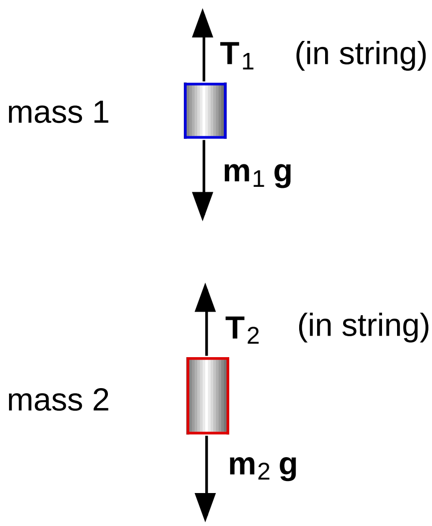
Solution: Assume acceleration at $\mathbf{m}_1$ is ↑ (which makes angular acceleration $\alpha > 0$ for counter-clockwise motion). Construct free-body diagrams:



Positive α corresponds to **a** upward at $\mathbf{m}_1$

$$|\mathbf{T}_1| = \mathbf{m}_1\mathbf{g} + \mathbf{m}_1\mathbf{a}$$

(assume **a** is upward)

$$|\mathbf{T}_2| = \mathbf{m}_2\mathbf{g} - \mathbf{m}_2\mathbf{a}$$

(acceleration in opposite direction to mass 1)

If there the string slides over the pulley without causing it to accelerate, the tension is constant along the string and the two equations can be solved to give **a**:

$$\mathbf{m}_1\mathbf{g} + \mathbf{m}_1\mathbf{a} = \mathbf{m}_2\mathbf{g} - \mathbf{m}_2\mathbf{a}$$

$$\mathbf{m}_1\mathbf{a} + \mathbf{m}_2\mathbf{a} = \mathbf{m}_2\mathbf{g} - \mathbf{m}_1\mathbf{g}$$

$$\mathbf{a}(\mathbf{m}_1 + \mathbf{m}_2) = \mathbf{g}(\mathbf{m}_2 - \mathbf{m}_1)$$

or

$$\mathbf{a} = (\mathbf{m}_2 - \mathbf{m}_1)/(\mathbf{m}_1 + \mathbf{m}_2)\ \mathbf{g},$$

which tells us that **a** is positive when $\mathbf{m}_2 > \mathbf{m}_1$, as expected.

In general:

around pulley axis: $\tau_1 = -\mathbf{R}\ \mathbf{T}_1$ (into plane)

$\tau_2 = \mathbf{R}\ \mathbf{T}_2$ (out of plane)

$$\tau_{net} = \mathbf{R}\,(-\mathbf{T}_1 + \mathbf{T}_2) = -\mathbf{R}\,\mathbf{g}\ (\mathbf{m}_1 - \mathbf{m}_2) - \mathbf{R}\ \mathbf{a}\ (\mathbf{m}_1 + \mathbf{m}_2)$$

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But τ gives an angular acceleration $\alpha = \frac{\tau}{I}$ to the pulley. By our convention,
a is ↑ α is  (*i.e.*, greater than zero).

$$I\,\alpha = -\mathbf{R}\,[\,\mathbf{g}\,(\mathbf{m}_1 - \mathbf{m}_2) + \mathbf{a}\,(\mathbf{m}_1 + \mathbf{m}_2)\,]$$

Lastly, since $\alpha\,\mathbf{R} = \mathbf{a}$, then

$$I\,\frac{\mathbf{a}}{\mathbf{R}} = -\mathbf{R}\,[\,\mathbf{g}\,(\mathbf{m}_1 - \mathbf{m}_2) + \mathbf{a}\,(\mathbf{m}_1 + \mathbf{m}_2)\,]$$

$$\mathbf{a}\,[\,\frac{I}{\mathbf{R}^2} + (\mathbf{m}_1 + \mathbf{m}_2)\,] = -\mathbf{g}\,(\mathbf{m}_1 - \mathbf{m}_2)$$

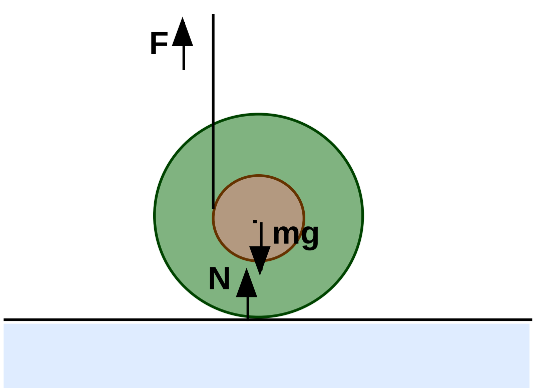
$$\mathbf{a} = \mathbf{g}\,\frac{\mathbf{m}_2 - \mathbf{m}_1}{\frac{I}{\mathbf{R}^2} + (\mathbf{m}_1 + \mathbf{m}_2)}$$

So, if $\mathbf{m}_2 > \mathbf{m}_1$, $\alpha > 0$ (counterclockwise)
$\mathbf{a} > 0$ (up at $\mathbf{m}_1$)

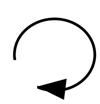
What happens if $I = 0$*?* Is the above expression for **a**, which used $\tau = I\,\alpha$, still valid? Yes, as can be seen by comparing with the expression for **a** on page 2.

<u>Example 2: yo - yo on a table</u>:

String held vertically as force is applied:



Only torque around centre is **Fr** giving a motion of



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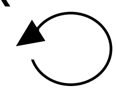
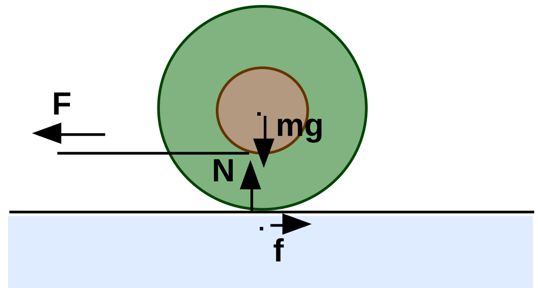
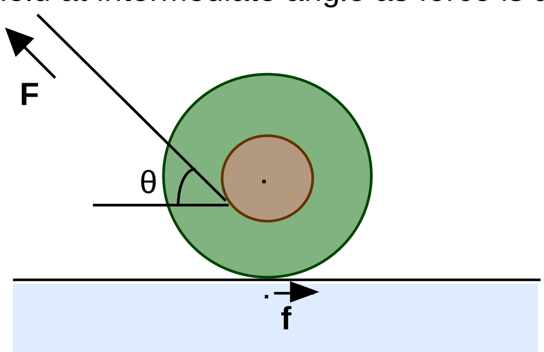
String held horizontally as force is applied:

When the acceleration is very small,
F   **f** (in magnitude).
    there is a net torque around the centre
        Fr < **fR**
    motion is 



String held at intermediate angle as force is applied:



At some point

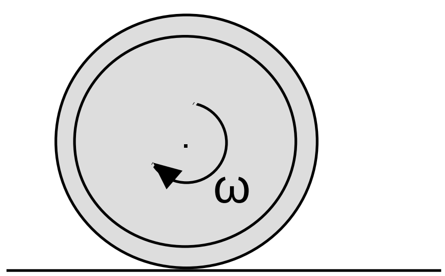
$$\mathbf{Fr} = \mu \mathbf{NR}$$

and there will be no torque.

At this angle, the yo-yo just slides along the table without rotating.

Rolling without Slipping

Consider the motion of a wheel for which there is no slippage between the wheel and the ground. If the wheel moves at an angular frequency ω around its axle,



then what are the velocities of the wheel's edge with respect to the ground? According to an observer moving at the same speed as the axle, the speed of the axle is zero and the velocities at the rim of the wheel are:

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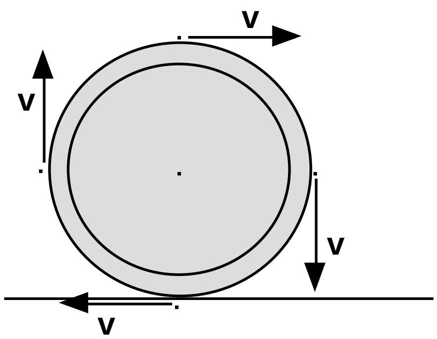
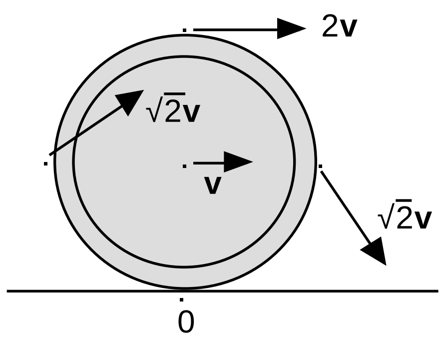


All points have the same speed, but different orientations for $\vec{\mathbf{v}}$.

According to an observer stationary with respect to the ground, add $\vec{\mathbf{v}}$ to each velocity above:



$$\vec{\mathbf{v}}_{ground} = \vec{\mathbf{v}} + \vec{\mathbf{v}}_{axle}$$

orientation from

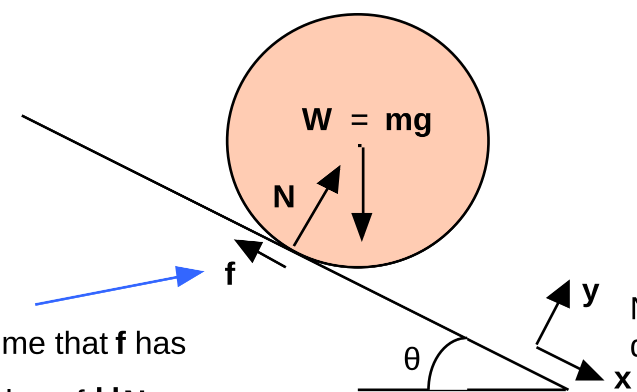
$$\vec{\mathbf{v}} = \vec{\omega} \times \vec{\mathbf{r}}$$

In magnitude,

$$\vec{\mathbf{v}}_{axle} = \vec{\mathbf{v}}_{wheel}$$

Wheel Rolling Down Incline Plane

The free - body diagram for a wheel on a plane looks like



Note axis directions

DO NOT assume that **f** has its maximal value of μ**N**

The effect of the frictional force is to produce a torque about the axle.

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x - direction: (along the plane)

$$m a_{cm} = mg \sin\theta - f \quad (1)$$

$$\tau = \vec{R} \times \vec{F} = -Rf$$

$$I\alpha = -Rf \qquad \text{from } \tau = I\alpha$$

Since the tangential acceleration on the rim is equal to a_{cm}, in the wheel's reference frame, then

$$a_{cm} = a_{tan} = \vec{\alpha} \times \vec{R} = -\alpha R$$

($\alpha > 0$ corresponds to uphill a_{cm})

$$I\left(\frac{-a_{cm}}{R}\right) = -Rf$$

or

$$a_{cm} = \frac{R^2}{I} f \quad (2)$$

Substituting the frictional force from (1) into (2) gives

$$a_{cm} = \frac{R^2}{I}(mg \sin\theta - m a_{cm})$$

$$a_{cm}\left(1 + \frac{mR^2}{I}\right) = \frac{mR^2}{I} g \sin\theta$$

?

$$a_{cm} = \frac{g \sin\theta}{1 + \dfrac{I}{mR^2}}$$

Why not just use $f = \mu N$?

Because frictional force is less than μN. If $\theta \sim 0$, $f \sim 0$ as well.

This tells us that a_{cm} *decreases* with increasing I for **M , R** fixed.

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Examples of rolling objects

Ring	Cylinder	Sphere
$I = \mathbf{MR}^2$	$I = \frac{1}{2}\mathbf{MR}^2$	$I = \frac{2}{5}\mathbf{MR}^2$
$\mathbf{a}_{cm} = \frac{1}{2}\mathbf{g}\sin\theta$	$\mathbf{a}_{cm} = \frac{2}{3}\mathbf{g}\sin\theta$	$\mathbf{a}_{cm} = \frac{5}{7}\mathbf{g}\sin\theta$
(slowest)		(fastest)

Note: the frictional force **f** can be calculated once $\mathbf{a}_{cm}$ is known, and can be compared against $\mu\mathbf{N}$ for consistency.

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