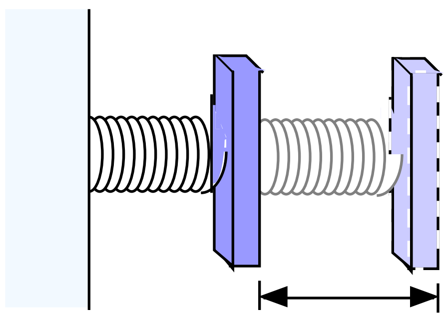
Demonstrations:
•mass on a spring, pendulum, stop clock
Text: Fishbane 13-1, 13-3, 13-4, 13-5
Problems: 13, 15, 43, 45, 49 from Ch. 13

What's important:
•simple harmonic motion

Oscillatory Motion

We introduced Hooke's Law as an example of a force that increases linearly with the displacement from equilibrium **x**:



$$\vec{\mathbf{F}} = -k\vec{\mathbf{x}}$$

The motion which an object executes under Hooke's Law as it oscillates back and forth is referred to as simple harmonic motion (or SHM). We want to find the displacement **x** as a function of time **x**(t). From Newton's Law **F** = m **a**, we have

$$m\,\mathbf{a}(\mathbf{t}) = -k\,\mathbf{x}(\mathbf{t}).$$

But $\mathbf{a} = d^2\mathbf{x} / dt^2$

so that $$d^2\mathbf{x} / dt^2 = -(k / m)\,\mathbf{x} \qquad (1)$$

Equation (1) is called a differential equation. It does not specify the form for **x**(t), in the sense that **x**(t) is on the left hand side of the equaiton, and a function is on the right hand side. Instead, it relates **x**(t) to its second derivative. This relation turns out to be sufficient to determine the functional form of **x**(t), although it is not sufficient to determine some of the constants in the function.

Clearly, functions like $x(t) = c\,t^n$ or $x(t) = c\exp(-bt)$ do not satisfy Eq. (1), as can be seen by explicit substitution. Physically, these functions decay to zero at large t, whereas what we want is a solution that oscillates. Let's try a trig function, which we know is oscillatory

$$\mathbf{x}(t) = A \sin \omega t$$

The proof is by direct substitution. Taking the first derivative:

$$dx / dt = d[A \sin \omega t] / dt = A \bullet d(\omega t)/dt \bullet d \sin q / dq \qquad \text{(where } q = \omega t\text{)}$$

$$= A\omega \bullet \cos q = A\omega \cos\omega t.$$

Taking the second derivative:

$$d^2x / dt^2 = d[A\omega \cos\omega t] / dt = A\omega \bullet d(\omega t)/dt \bullet d \cos q / dq \qquad \text{(where } q = \omega t\text{)}$$

$$= A\omega\omega \bullet [-\sin q] = -A\omega^2 \bullet \sin\omega t.$$

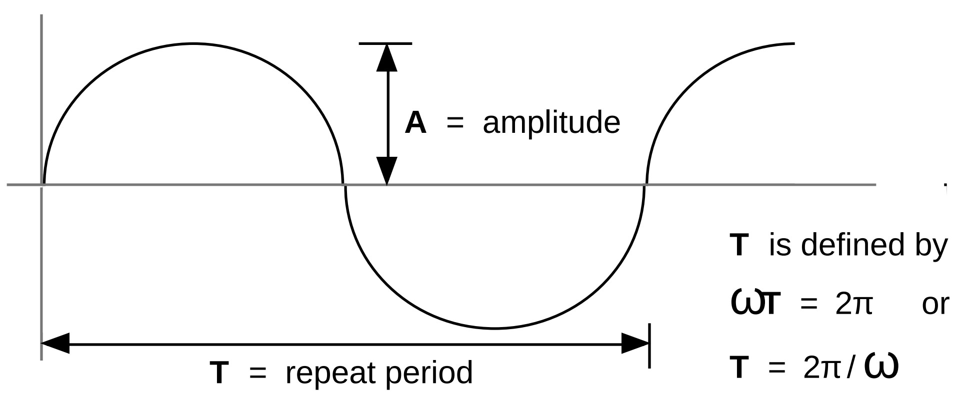
Replacing $A \sin\omega t$. with $x(t)$, we have just shown that

$$d^2x / dt^2 = -\omega^2 x(t).$$

This is the form required for simple harmonic motion, and we have now established that the angular frequency ω is given by

$$\omega^2 = k / m \qquad \text{or} \qquad \omega = (k / m)^{1/2}$$

A graph of **x**(**t**) looks approximately like (my drawing routines unfortunately don't produce trig functions!):



Substituting $\omega = (k / m)^{1/2}$ into the expression for the period T gives

$T = 2\pi (m / k)^{1/2}$ or $f = (1 / 2\pi) (k / m)^{1/2}$ ($f = 1/T$ = frequency)

The form $\mathbf{x}(t) = A \sin\omega t$ satisfies $\mathbf{x} = 0$ at $t = 0$. A more general solution, which allows $\mathbf{x}$ to be non-zero at $t = 0$ is $\mathbf{x}(t) = A \sin(\omega t + \delta)$, where δ is a so-called phase angle. If we wish to impose $x = A$ at $t = 0$, then we can use $\mathbf{x}(t) = A \cos\omega t$.

Energy Conservation in SHM

Let's now calculate the kinetic and potential energy in simple harmonic motion. The potential energy is easy:

$$U = \frac{1}{2} kx^2 = \frac{1}{2} kA^2 \sin^2 \omega t$$

For the kinetic energy, we need **v**(**t**), which we have obtained above

$$\mathbf{v}(t) = d\mathbf{x}(t) / dt = A\omega \cos\omega t.$$

Hence $K = mv^2/2 = m[A\omega \cos\omega t]^2 / 2 = mA^2\omega^2 \cos^2\omega t / 2$.

Adding the kinetic and potential energies gives

$$E = K + U = mA^2\omega^2 \cos^2\omega t / 2 + kA^2 \sin^2\omega t / 2$$

$$= mA^2 [k / m] \cos^2\omega t / 2 + kA^2 \sin^2\omega t / 2$$

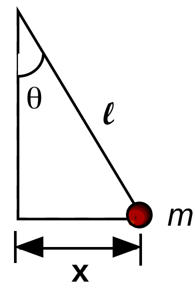
$$= kA^2 [\cos^2\omega t + \sin^2\omega t] / 2$$

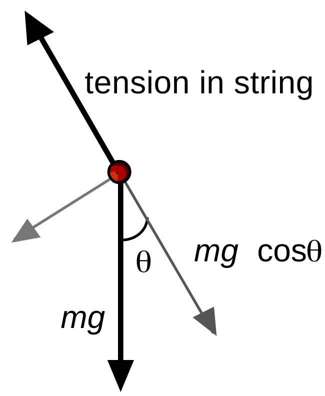
$$= kA^2 / 2.$$

Since k and A are both constant, the sum $K + U$ is also a constant, and energy is conserved in the motion.

Simple Pendulum

Consider a mass m suspended by a massless string of length ℓ. If we move the mass away from its equilibrium position, then it is subject to a restoring force





In a coordinate system which has one axis along the string,

tension in string = $mg \cos\theta$	balanced
$F_R = -mg \sin\theta$	unbalanced

But $\sin\theta = \dfrac{x}{\ell}$, so

$$F_R = -mg\frac{x}{\ell}$$

For θ small, F_R is roughly horizontal, so

$$F_R = ma = -mg\frac{x}{\ell} \quad \text{or} \quad a = -\frac{g}{\ell}x$$

($\vec{\mathbf{a}}$ is in opposite direction to $\mathbf{x}$)

The relation between *a* and *x* is that of simple harmonic motion, so

$$\omega = \sqrt{\frac{g}{\ell}} \qquad \text{or} \qquad T = 2\pi \sqrt{\frac{\ell}{g}}$$

Note: *T* does *not* depend on *m* or *A*.

Example

What is the period of a pendulum 1.00 m long?

Solution:

$$T = 2\pi \sqrt{\frac{1.00}{9.81}} = 2.00_6 \text{ sec.}$$

A pendulum with a period of exactly 2 sec is referred to as a seconds pendulum.

Demo:

•Simple pendulum 1 m in length; measure period with a stop clock; ~2 secs.

•Compare periods of pendula with same length, different masses; observe same periods.