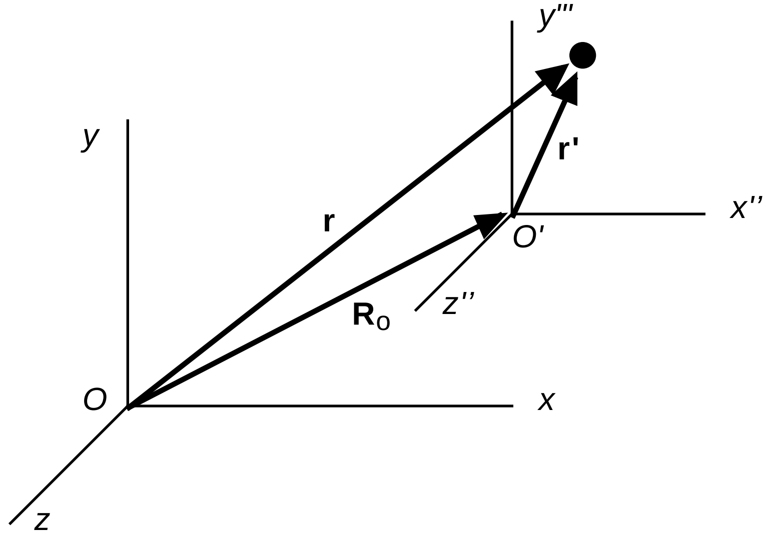
Accelerated coordinate systems

Most of our intuition about mechanics comes from our experience on Earth: we describe projectile motion, spring oscillations *etc.* with respect to a coordinate system that we call “at rest”. But at rest with respect to what? We know that the surface of the is not only moving, it is accelerating! Further, the Earth itself is accelerating around the Sun, and the whole solar system is accelerating around the Milky Way. What affect does a translating or rotating coordinate system have on the apparent laws of mechanics? In the next four lectures, we consider several situations

(i) non-rotating coordinate systems and the Galilean transformation
(ii) space-time and the Lorentz transformation
(iii) rotating coordinate systems
(iv) rotation of the Earth

Non-rotating coordinate systems

Consider two coordinate systems (x , y , z) and (x' , y' , z') whose origins lie at O and O' respectively. O' has position $\mathbf{R}_0$ with respect to O , and $\mathbf{R}_0$ may be a function of time



We may alternatively refer to the different coordinate systems as different reference frames. The system *at rest* is referred to as the inertial system.

From the diagram, it is clear that an object at position $\mathbf{r}$ in system O will be labelled by $\mathbf{r}'$ in O', with

$$\mathbf{r} = \mathbf{R}_0 + \mathbf{r}'$$

What happens if the object is moving? Then the velocities are related by

$$d\mathbf{r} / dt = d\mathbf{R}_O / dt + d\mathbf{r} / dt$$

or equivalently

$$\mathbf{v} = \mathbf{V}_O + \mathbf{v}'$$

where $\mathbf{V}_O$ is the velocity of coordinate origin *O′*

$$\mathbf{V}_O = d\mathbf{R}_O / dt$$

Similarly, one can take another derivative to obtain a relationship between the accelerations in the two frames

$$\mathbf{a} = \mathbf{A}_O + \mathbf{a}',$$

where $\mathbf{A}_O = d\mathbf{V}_O / dt$.

Now, if *O′* is not accelerating with respect to *O*, then $\mathbf{A}_O = 0$ and $\mathbf{a} = \mathbf{a}'$.

Thus, if there is a given force $\mathbf{F}$ acting on an object, the acceleration that it produces will be the same in both frames: $\mathbf{F} = m\mathbf{a} = m\mathbf{a}'$. Equivalently, Newtons's law will be the same in both frames.

However, if $\mathbf{A}_O$ is not zero, then the dynamics of a particle's motion will appear to be different

$$\mathbf{F} = m\mathbf{a} = m\mathbf{A}_O + m\mathbf{a}'.$$

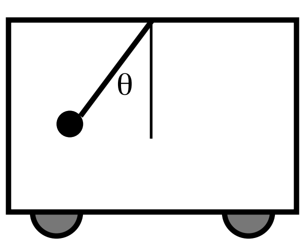
So, the apparent force $\mathbf{F}'$ in frame *O′* will be

$$\mathbf{F}' = m\mathbf{a}' = \mathbf{F} - m\mathbf{A}_O \qquad (1)$$

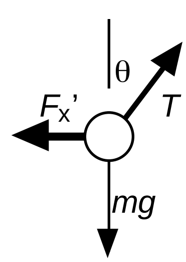
That is, an observer in *O′* will add a fictitious force $m\mathbf{A}_O$ to the known force $\mathbf{F}$ in order to make Newton's laws work.

<u>Example</u>

Consider the position of a mass hanging from a string in an accelerating subway car.



Free-body diagram in *O′*



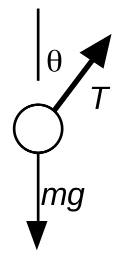
To an observer on the car:
m does not accelerate, hence $\sum \mathbf{F}' = 0$ according to Newton. To account for the x - component of the tension, an observer in the car would say:

$$T\cos\theta = mg$$
$$F_x' = -T\sin\theta$$
$$\Rightarrow \quad F_x' = -mg\tan\theta \qquad (2)$$

So, the accelerating observer claims that there is a force $F_x' = -mg\tan\theta$ acting on the mass.

An inertial observer (on the ground looking at the subway car) would see that the mass is undergoing an acceleration $\mathbf{A}_o$ because of the *unbalanced* tension in the string

Free-body diagram in *O*



$$T\cos\theta = mg$$
$$T\sin\theta = mA_o$$
$$\Rightarrow \quad A_o = g\tan\theta \qquad (3)$$

These two views can be resolved, since (2) + (3) implies that (removing $g\tan\theta$)

$$F_x' = -m\mathbf{A}_o$$

This is just what one expects from Eq.(1) if $\mathbf{F} = O$

<u>Galilean transformation</u>

The situation in which $\mathbf{A}_o$ is not only a constant, but vanishes, is called a Galilean transformation. In one dimension it reads

$$x' = x - X_o$$
$$v' = v - V_o$$
$$a' = a$$

But the position of the moving coordinate origin is $X_0 = V_0 t$, so

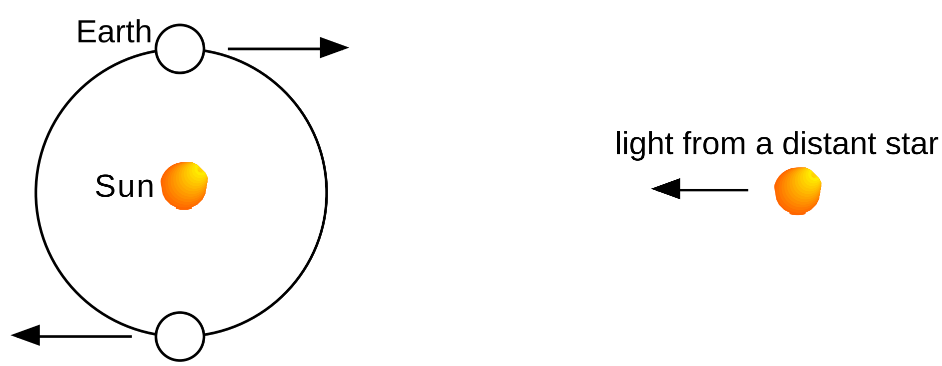
$$x' = x - V_0 t$$
$$t' = t$$

where we have added the implicit asumption that time is *not* motion-dependent. The fact that Newton's Laws are invariant under a Galilean transformation gives us some confidence that it is correct.

This implicit view of space-time was fine for 1670, but....

1. By 1864 James Clerk Maxwell had written down a complete set of equations (Maxwell's equations) to describe the motion of charged objects in electric and magnetic fields. But Maxwell's equations changed under a Galilean transformation! Who was wrong: Maxwell or Galileo?

2. Maxwell's equations predicted the existence of electromagnetic waves, including their speed *c*. The predicted value of *c* was very close to the observed value of the speed of light. Experiments were then performed to measure *c* very accurately and
   i) compare it with Maxwell's predictions
   ii) see how it varied with the speed of the Earth.



They found that the speed of light did not change with the motion of the Earth. [Michelson and Morley (1881 and 1887)], directly contradicting the Galilean transformation.