Lorentz transformation

In the early 1900's, Einstein began to ponder the properties of Maxwell's equations, in particular the problem of Galilean invariance. He made two postulates:

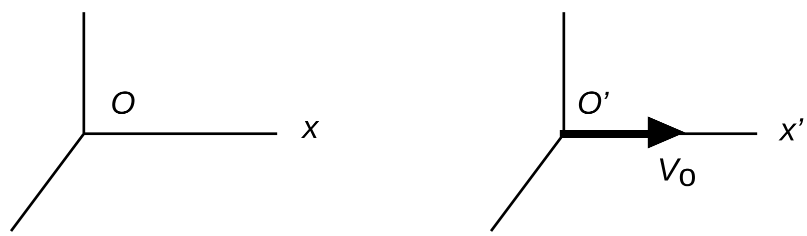
1. absolute uniform motion cannot be detected
2. the speed of light is independent of the motion of the source

But Einstein was forced to give up the Galilean transformation since it implied that the measured speed of light c was frame-dependent

$$c' = c - V_o$$

Instead, he "rediscovered" and reinterpreted a new transformation called the Lorentz transformation in honour of H. A. Lorentz, who had discovered it while trying to understand the motion of charged object in the hypothetic "ether" which supported electric and magnetic fields. The derivation of the transformation goes as follows:

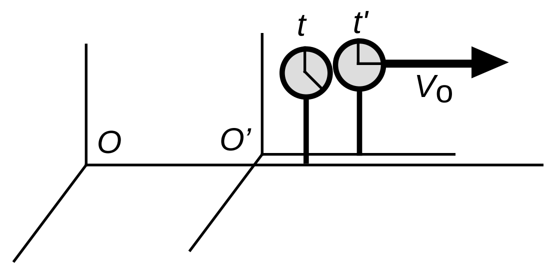
We start with two coordinate systems O , O' whose x - axes are collinear. The velocity V_o is assumed to be along the x - axis



We **DROP** the assumption that time has the same value as both frames: $t \neq t'$

Now, consider a measurement of the speed of light. Let frame O be the ground, and O' be a train moving along a long straight track. When O and O' are coincident, we set $t = t' = 0$. Then the position of O' as seen by O is just $V_o t$.

A flash of light goes off at $t = t' = 0$. Sometime later, it illuminates two clocks, one on O and one on the train moving at a speed V_o.



(For example, an observer on the train O' at x' could be holding a stopwatch which he reads when the stationary clock at x is illuminated by the flash originating at $t = t' = 0$. Then t' is the time on his stop watch, and t is the time on the clock illuminated by the flash.)

In the two frames, the speed of light, distance over time, is then

$$c = (x^2 + y^2 + z^2)^{1/2} / t \qquad \text{in } O \qquad (1a)$$
$$c' = (x'^2 + y'^2 + z'^2)^{1/2} / t' \qquad \text{in } O' \qquad (1b)$$

Rewriting this, and throwing away the y and z coordinates:

$$x^2 - c^2t^2 = 0 \qquad (2a)$$
$$x'^2 - c'^2t'^2 = 0 \qquad (2b)$$

We demand $c = c'$, which in turn says that the combination $x^2 - c^2t^2$ must be frame-independent. What transformation will do the job?

The correct transformation between x, t and x', t' must be linear in the quantities, or there would be 2 or more values of x' for a given x [when one finds the roots of the polynomial]. Thus, we try a relationship like

$$x' = \gamma x + \delta t \qquad \gamma, \delta \text{ are constants} \qquad (3)$$

But we know that the origin of O', namely $x' = O$, is described by $V_o t$, that is $x' = \gamma x + \delta y$ becomes

$$0 = \gamma V_o t + \delta t$$
or $$\delta = -\gamma V_o \qquad (4)$$

Thus, we try the linear relations

$$x' = \gamma \, (x - V_o t) \qquad (5)$$
$$x = \gamma' \, (x' + V_o t') \qquad (6)$$

Our task is to find γ . We return to Eqs (2a) and (2b), combining them as

$$x^2 - c^2t^2 = x'^2 - c^2t'^2 \qquad (7)$$

To get rid of x' and t' in (7) we use Eq. (5) for x' , and obtain a further relation for t' using (5) + (6):

$$x = \gamma' [\gamma (x - V_o t) + V_o t]$$

=> $$x = \gamma\gamma' (x - V_o t) + \gamma' V_o t'$$

or

$$t' = [x (1-\gamma\gamma') + \gamma\gamma' V_o t] / \gamma' V_o \qquad (8)$$

Now we can put (5) and (8) into (7)

$$x^2 - c^2 t^2 = [\gamma (x - V_o t)]^2 - c^2 [x (1-\gamma\gamma') + \gamma\gamma' V_o t]^2 / (\gamma' V_o)^2$$

$$= \gamma^2 (x^2 - 2V_o xt + V_o^2 t^2) - c^2 [x^2 (1-\gamma\gamma')^2 + 2x (1-\gamma\gamma')\gamma\gamma' V_o t + (\gamma\gamma' V_o t)^2] / (\gamma' V_o)^2$$

Collecting all of the terms, we find

$$x^2[1- \gamma^2 + (c^2\gamma^2 / V_o^2)(1/\gamma\gamma' - 1)]$$
$$+ xt [2\gamma^2 u + 2c^2\gamma^2 (1/\gamma\gamma' - 1)/V_o]$$
$$+ t^2 (c^2\gamma^2 - c^2 - \gamma^2 V_o^2) = 0 \qquad (9)$$

Now, this expression must be valid for any arbitrary x,t . Hence, all of the coefficients must vanish. Starting with the coefficient for t^2:

$$c^2\gamma^2 - c^2 - \gamma^2 V_o^2 = 0$$
$$\gamma^2(c^2 - V_o^2) = c^2$$
$$\gamma^2 = 1 / (1- V_o^2/c^2)$$

or finally $$\gamma = 1 / (1- V_o^2/c^2)^{1/2} \qquad (10)$$

If we substitue (10) back into the xt coefficient in Eq. (9) we find $\gamma' = \gamma$ (left as an exercise). We can now use γ, γ' in Eqs. (5) and (6) for x, x' . To obtain the time transformations, we substitute into Eq. (8); first using $\gamma = \gamma'$

$$t' = [x(1-\gamma^2) + \gamma^2 V_o t] / \gamma V_o$$
$$= (\gamma^2 / \gamma V_o)[x(1 / \gamma^2 - 1) + V_o t]$$
$$= \gamma [(x /V_o)(1 - V_o^2 / c^2 - 1) + t]$$
$$= \gamma [t - xV_o / c^2]$$

In summary:

$$x' = \gamma\,(x - V_o t) \qquad x = \gamma\,(x' + V_o t')$$
$$t' = \gamma\,[t - (V_o x / c^2)] \qquad t = \gamma\,[t' + (V_o x' / c^2)]$$

Note the very obvious conclusion of the Lorentz transformation: $t \neq t'$ and there is no absolute time.