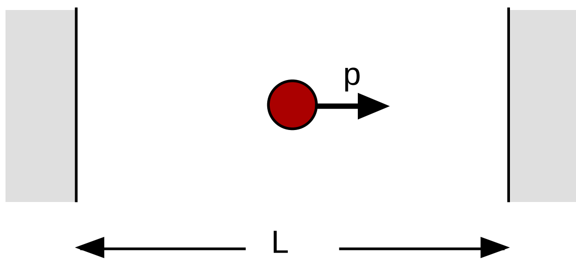
We do very little quantum mechanics in PHYS 120 beyond introducing the de Broglie wavelength $\lambda = h/\mathbf{p}$. Here, we explore the properties of one of the simplest quantum systems, namely a particle confined by infinitely hard walls. Our results are applied in the next lecture to neutron stars.

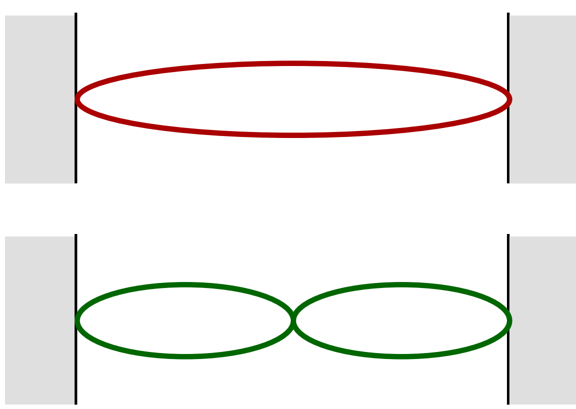
Particle in a 1D square well

The illustration shows a very simple system - a particle moving in one dimension between two infinitely hard walls.



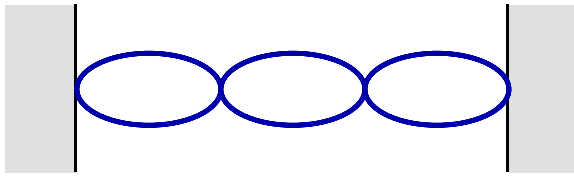
Since the walls are infinitely hard, then the particle bounces off each wall with no change in the magnitude of its momentum: it always has the same momentum **p** and de Broglie wavelength λ.

We know from the properties of a guitar or piano string, that a given string supports only certain well-defined frequencies or wavelengths. That is, if we pluck on a string tuned to the chromatic note A, we hear only A (and its harmonics), not an arbitrary note. Looking at the string, we find that the "allowed wavelengths" of the string are related to its length as follows



$L = 1/2\ \lambda$

$L = 2/2\ \lambda$



$L = 3/2\ \lambda$

The ellipses in the diagram are supposed to represent the maximum displacements of the string from equilibrium. In a guitar string, other waves may be present when the string is initially plucked, but they die away very rapidly.

Applying this "standing wave" idea to the motion of a particle between hard walls, we conclude that there is a specific relationship between the de Broglie wavelength and the distance between the reflecting walls:

$$\mathbf{L} = (\mathbf{n}/2)\ \lambda \qquad \text{where } \mathbf{n} = 1, 2, 3, 4...$$

This relationship says that the wavelengths are "quantized": only certain specific wavelengths are allowed for the motion. Similarly, there are a series of allowed values for the momentum

$$\mathbf{p}_n = h/\lambda = \mathbf{n}h / 2\mathbf{L} \qquad \text{where } \mathbf{n} = 1, 2, 3, 4....$$

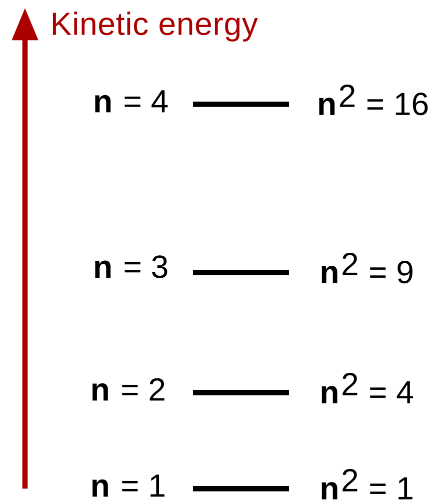
and the kinetic energy

$$\mathbf{K}_n = \mathbf{p}^2 / 2\mathbf{m} = (1 / 2\mathbf{m})\ (\mathbf{n}h / 2\mathbf{L})^2$$

or

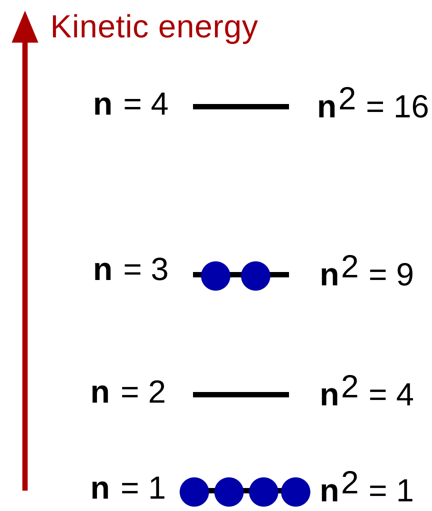
$$\mathbf{K}_n = \mathbf{n}^2 h^2 / 8m\mathbf{L}^2 \qquad \text{where } \mathbf{n} = 1, 2, 3, 4....$$

Both the momentum and kinetic energy are quantized, and characterized by a quantum number **n** (which is used as a subscript on $\mathbf{K}_n$ and $\mathbf{p}_n$). The allowed energies have a spectrum that looks like

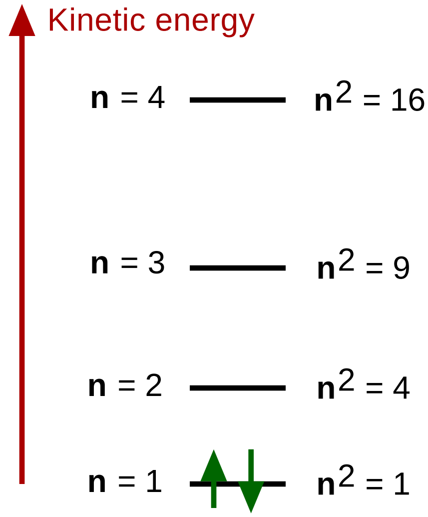


A single particle, for example, an electron, can have any of these energies. However, when we place many particles into the region between the walls, there may be restrictions on what energies they may have *even if the particles have no explicit interactions among them*.

Bosons (for example, photons or pions): there are no restrictions on how many bosons can have the same energy



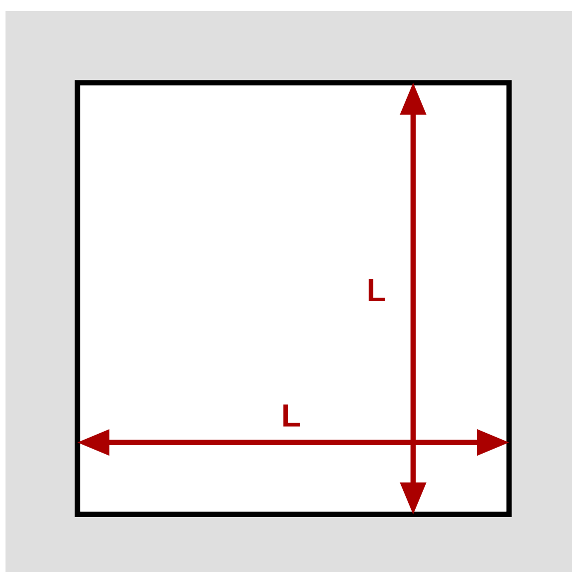
Fermions (for example, electrons and protons)



No two fermions can have all of their quantum numbers the same. In the diagram, there can be at most two particles such as electrons, protons or neutrons in the same energy level, since each particles can have its spin pointing in different directions.

Particle in a 3D box

Consider the motion of a particle in a three-dimensional hard box; that is, a cubical box with infinitely hard walls. A two-dimensional slice of this box would look like



A single particle can move in any of the three directions independently, and its momentum in each of the directions is quantized, as in the one-dimensional case:

$$\mathbf{p}_x = \mathbf{n}_x h / 2\mathbf{L} \qquad \text{where } \mathbf{n}_x = 1, 2, 3, 4....$$

$$\mathbf{p}_y = \mathbf{n}_y h / 2\mathbf{L} \qquad \text{where } \mathbf{n}_y = 1, 2, 3, 4....$$

$$\mathbf{p}_z = \mathbf{n}_z h / 2\mathbf{L} \qquad \text{where } \mathbf{n}_z = 1, 2, 3, 4....$$

The kinetic energy **K** is a scalar, of course, and a sum of the individual components of momentum

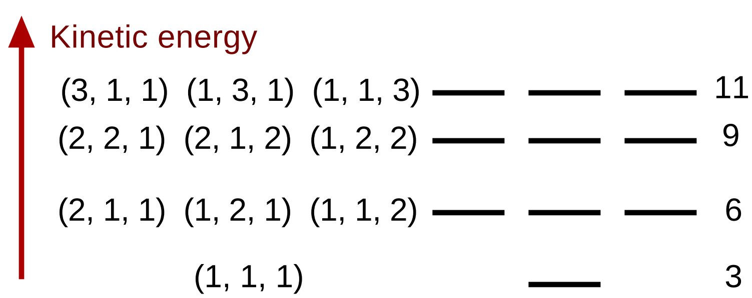
$$\mathbf{K} = (1 / 2\mathbf{m})\,(\mathbf{p}_x^2 + \mathbf{p}_y^2 + \mathbf{p}_z^2)$$

$$= (h^2 / 8m\mathbf{L}^2)\,(\mathbf{n}_x^2 + \mathbf{n}_y^2 + \mathbf{n}_z^2)$$

where $\mathbf{n}_x$ = 1, 2, 3, 4.... $\mathbf{n}_y$ = 1, 2, 3, 4.... $\mathbf{n}_z$ = 1, 2, 3, 4....

Note:
(i) Kinetic energy does not have components: no $\mathbf{K}_x$, $\mathbf{K}_y$, $\mathbf{K}_z$.
(ii) All **n**'s must be greater than 0 or the wavefunction vanishes.

The energy level scheme is now somewhat more complicated than the one-dimensional case:



We note two immediate differences with the 1 dimensional system:

(i) the energy levels are more closely spaced in 3D

(ii) several different combinations of ($\mathbf{n}_x$, $\mathbf{n}_y$, $\mathbf{n}_z$) may have the same energy. For example, (1, 1, 2) (1, 2, 1) and (2, 1, 1) all have $\mathbf{n}_x^2 + \mathbf{n}_y^2 + \mathbf{n}_z^2 = 6$, and hence all have the same energy.

Different configurations with the same energy are said to be *degenerate*.

When the box contains many partilces, they occupy energy levels according to the same rules as the one-dimensional situation. Fermions, for example: