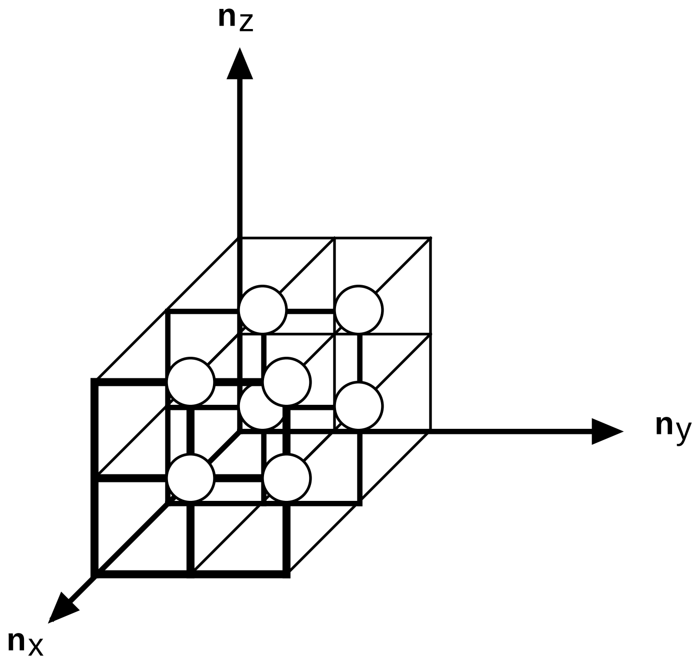
Ground state of many-fermion systems

In the previous lecture, we showed that the energy levels for a particle in a three-dimensional cubical box obeyed

$$\mathbf{K} = (1 / 2\mathbf{m})\ (\mathbf{p}_x^2 + \mathbf{p}_y^2 + \mathbf{p}_z^2)$$

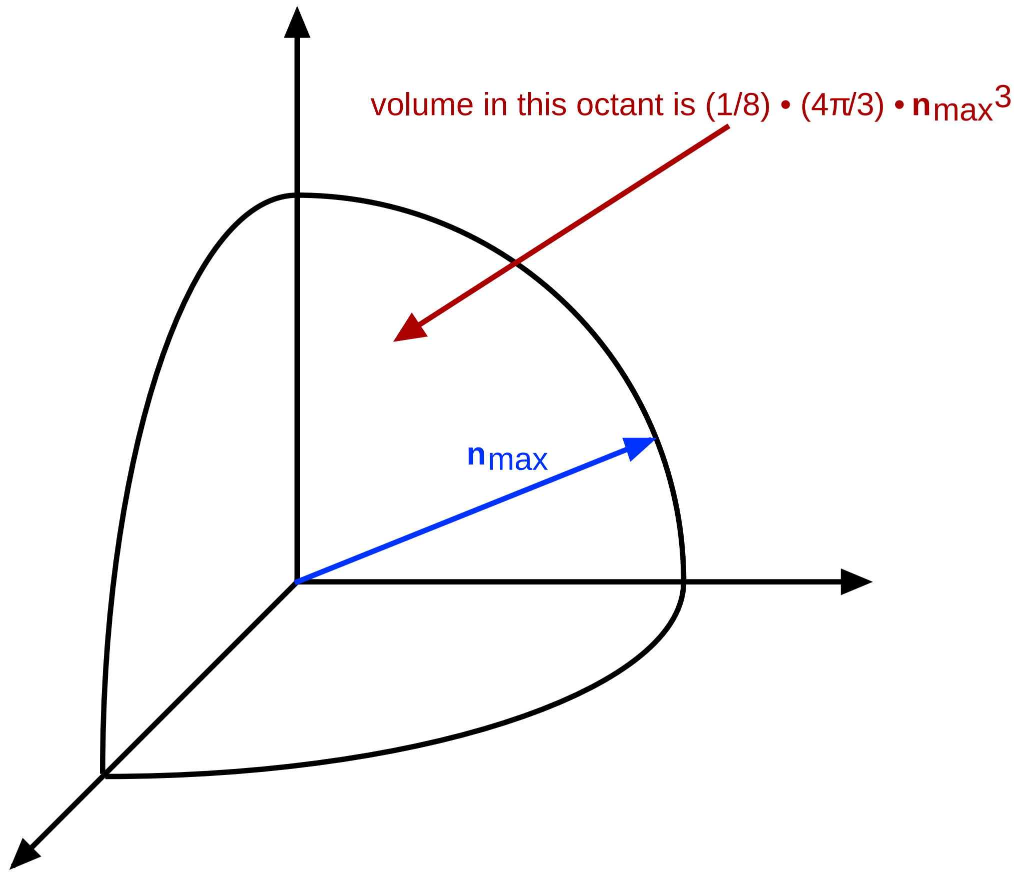
$$= (h^2 / 8\mathbf{mL}^2)\ (\mathbf{n}_x^2 + \mathbf{n}_y^2 + \mathbf{n}_z^2)$$

where $\mathbf{n}_x = 1, 2, 3, 4....$ $\mathbf{n}_y = 1, 2, 3, 4....$ $\mathbf{n}_z = 1, 2, 3, 4....$ That is, for every positive integer value for $\mathbf{n}_x$, $\mathbf{n}_y$ and $\mathbf{n}_z$, there is an allowed energy state. Plotting the values of ($\mathbf{n}_x$, $\mathbf{n}_y$, $\mathbf{n}_z$) for which there are states, we find:



In the drawing, there are 8 allowed states, one for each elementary box of dimension 1 x 1 x 1. That is, the total number of unit boxes in the 2 x 2 x 2 cube is 8, and the number of allowed states is 8.

Consider now what happens with a large number of levels. The number of levels with a maximum value $\mathbf{n}_{max}$ for any quantum number can be found by the following argument:



Since there is one state for every box of unit volume, then the total number of states with **n**'s less than $\mathbf{n}_{max}$ is just the volume of the octant. That is, the number of states is

$$(1/8) \bullet (4\pi/3) \bullet \mathbf{n}_{max}^3 = (\pi/6)\, \mathbf{n}_{max}^3$$

Filling these states with electrons (or protons, or neutrons), the total number of fermions **N** is

$$\mathbf{N} = 2 \bullet (\pi/6)\, \mathbf{n}_{max}^3 = (\pi/3)\, \mathbf{n}_{max}^3$$

since there are two electrons allowed in every state.

What's so important about $\mathbf{n}_{max}$? Why do we care about this calculation? The quantity $\mathbf{n}_{max}$ acts like a radius, and defines the maximum value of $(\mathbf{n}_x^2 + \mathbf{n}_y^2 + \mathbf{n}_z^2)$, just like the radius of a solid sphere is the maximum displacement of an element of the sphere from its centre. Thus

$$(\mathbf{n}_x^2 + \mathbf{n}_y^2 + \mathbf{n}_z^2)^{1/2} < \mathbf{n}_{max}.$$

But the kinetic energy depends on $(\mathbf{n}_x^2 + \mathbf{n}_y^2 + \mathbf{n}_z^2)$, so that

$$\mathbf{K} = (h^2 / 8\mathbf{mL}^2)\,(\mathbf{n}_x^2 + \mathbf{n}_y^2 + \mathbf{n}_z^2) < (h^2 / 8\mathbf{mL}^2)\,\mathbf{n}_{max}^2.$$

This bound on $\mathbf{K}$ is called the *Fermi energy* or $\mathbf{E}_f$. Thus:

At large $\mathbf{N}$, the maximum kinetic energy of an $\mathbf{N}$-fermion system in its ground state is $\mathbf{E}_f$. By "ground state", we mean that all of the fermions are forced to lie in the lowest energy configurations available.

A simpler expression for $\mathbf{E}_f$ can be obtained by replacing $\mathbf{n}_{max}$ via

$$\mathbf{N} = (\pi/3)\,\mathbf{n}_{max}^3 \quad \text{or} \quad \mathbf{n}_{max} = (3\mathbf{N} / \pi)^{1/3}.$$

Then

$$\mathbf{E}_f = (h^2 / 8\mathbf{mL}^2)\,(3\mathbf{N} / \pi)^{2/3} = (h^2 / 8\mathbf{m}) \bullet (3 / \pi)^{2/3} \bullet (\mathbf{N} / \mathbf{L}^3)^{2/3}.$$

But $\mathbf{N} / \mathbf{L}^3$ is just the density of particles - the number $\mathbf{N}$ per unit volume $\mathbf{V} = \mathbf{L}^3$. So,

$$\mathbf{E}_f = (h^2 / 8\mathbf{m}) \bullet (3 / \pi)^{2/3} \bullet (\mathbf{N} / \mathbf{V})^{2/3}$$

Application: conduction electrons

In conductors such as copper, most electrons in each atom are tightly bound to its nucleus and only a few are sufficiently loosely bound that they can move relatively freely through the conducting material. The loosely-bound or *conducting* electrons move in a potential in which the strong charge of the atomic nucleus is *screened* by the many electrons bound by it. Thus, the particle-in-a-box problem makes a good starting point for describing such electrons.

Typically there are only one or two conducting electrons per atom in a conductor. As an illustrative example, suppose that there is one electron per atom, and the that atom has a radius **R** of 0.2 nm. As an approximation, assume that the volume taken up by each atom is $(4\pi / 3)\, \mathbf{R}^3$, or 0.034 nm^3.

The density of conducting electrons is then

$$\text{density} = 1 \text{ electron} / 0.034 \text{ nm}^3 = 30 \text{ nm}^{-3} = 3 \times 10^{28} \text{ m}^{-3}$$

In fact, typical metals have conducting-electron densities of 1 - 8 x 10^{28} m^{-3}.

The Fermi energy corresponding to this density is

$$\mathbf{E}_f = (h^2 / 8\mathbf{m}) \bullet (3 / \pi)^{2/3} \bullet (\mathbf{N} / \mathbf{V})^{2/3}$$

$$= \frac{(6.63 \times 10^{-34} \text{ J-s})^2}{8 \times 9.11 \times 10^{-31} \text{ kg}} \left(\frac{3}{\pi}\right)^{2/3} (3 \times 10^{28} \text{ m}^{-3})^{2/3}$$

$$= 5.7 \times 10^{-19} \text{ J}$$

Converting this result to electron-volts yields

$$\mathbf{E}_f = 3.5 \text{ eV.}$$

The electron velocity corresponding to this kinetic energy is

$$v = \left(\frac{2\,K}{m}\right)^{1/2} = \left(\frac{2 \bullet 5.7 \times 10^{-19}}{9.11 \times 10^{-31}}\right)^{1/2} = 1.1 \times 10^6 \text{ m/s}$$

or about 0.3% of the speed of light. Not bad for an electron at room temperature.

Application: electrons in the Sun

The temperature of the interior of stars is typically larger than 10^7 K, which is too hot for atoms to exist as bound states. Treating electrons as unbound particles traversing the Sun, we find that their Fermi energy is number of electrons in the Sun ~ 10^{57}

volume of Sun = $(4\pi / 3)\, \mathbf{R}_{sun}^3 = (4\pi / 3)\,(7 \times 10^8)^3 = 1.4 \times 10^{27}$ m^3

electron density = $10^{57} / 1.4 \times 10^{27} = 7 \times 10^{29}$ m^{-3}

$$\mathbf{E}_f = \frac{(6.63 \times 10^{-34} \text{ J-s})^2}{8 \times 9.11 \times 10^{-31} \text{ kg}} \left(\frac{3}{\pi}\right)^{2/3} (7 \times 10^{29} \text{ m}^{-3})^{2/3}$$

$$= 4.6 \times 10^{-19} \text{ J}$$

Thus, the Fermi energy is about 30 eV, or about ten times the Fermi energy of an electron in a conductor. Normally, we don't worry about the Fermi energy of electrons in stars, since the thermal energy scale k_BT is

$$k_BT = 1.38 \times 10^{-23} \bullet 10^7 \sim 1.4 \times 10^{-16} \text{ J} \sim 900 \text{ eV}.$$

Application: neutron stars

Let's consider the general problem of a star comprised of freely-moving electrons and protons, in which we assume that

•the number of electrons and protons is the same (i.e., the star is not charged)

•the Coulomb interaction among the particles can be ignored.

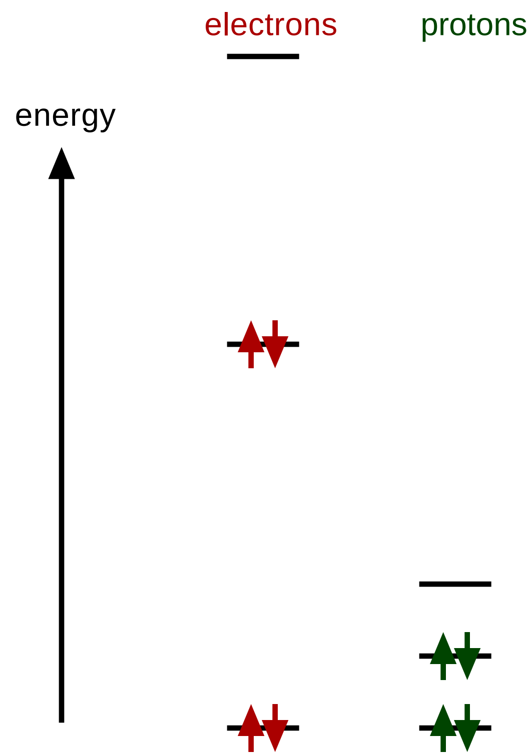
Then both the electrons and protons will have a set of energy levels which they occupy independently.

The separation between successive electron energy levels is larger than that of the proton energy levels because the Fermi energy is inversely proportional to the particle mass. Since the proton is 1830 times as massive as the electron, then the Fermi energy of a proton gas is 1 / 1830 times that of an electron gas at the same density. In the Sun, then

$$\mathbf{E}_f \text{ (electron)} \sim 30 \text{ eV}$$

$$\mathbf{E}_f \text{ (proton)} \sim 0.016 \text{ eV}.$$

Diagramatically,



The final stages of a star's life usually follow one of two scenarios:

- the nuclear fires in small stars go out, and the star collapses to a stable configuration
- the nuclear fires in large stars heat the star into the 10^8 K region and the star explodes.

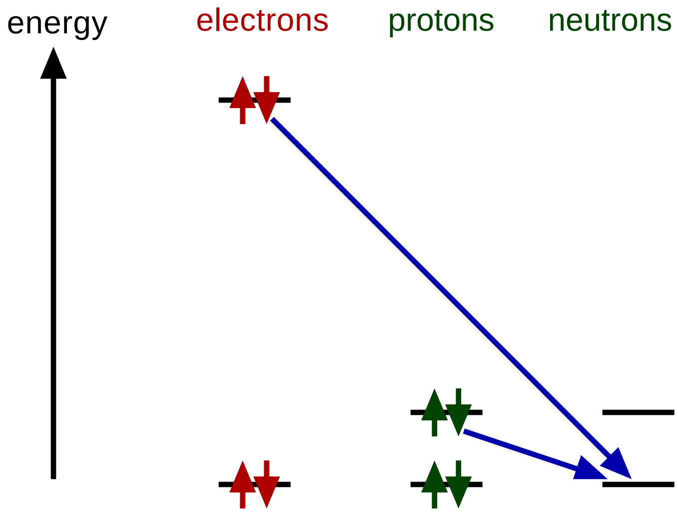
The cores of exploding stars can become very dense, and the corresponding Fermi energy of electrons can become very high. If the density is high enough, the Fermi energy can reach millions of eV.

At high density, one might find that

$$\mathbf{m}_e\mathbf{c}^2 + \mathbf{E}_f\,\text{(electron)} + \mathbf{m}_p\mathbf{c}^2 + \mathbf{E}_f\,\text{(proton)} \quad > \quad \mathbf{m}_n\mathbf{c}^2 + \mathbf{E}_f\,\text{(neutron)}$$

If this occurs, then it become energetically favourable for an electron and proton to react to form a neutron, releasing a neutrino:

$$e^- + p \rightarrow n + \nu.$$



At what density does this conversion to neutrons begin? The following calculation is a little crude, but it's good for an order of magnitude estimate. If we neglect the proton and neutron Fermi energies in the above expression for the threshold energy, then the threshold condition is

$$\mathbf{m_e c^2} + \mathbf{E_f}\,(\text{electron}) + \mathbf{m_p c^2} \;>\; \mathbf{m_n c^2}$$

or

$$\mathbf{E_f}\,(\text{electron}) \;>\; \mathbf{m_n c^2} - \mathbf{m_p c^2} - \mathbf{m_e c^2} = 939.566 - 938.272 - 0.511 \text{ (MeV)}$$

or

$$\mathbf{E_f}\,(\text{electron}) \;>\; 0.783 \text{ MeV} = 1.25 \times 10^{-13} \text{ J}$$

This corresponds to a density of

$$(\mathbf{N} / \mathbf{V})^{2/3} = \frac{8 \times 9.11 \times 10^{-31} \text{ kg}}{(6.63 \times 10^{-34} \text{ J-s})^2} \left(\frac{\pi}{3}\right)^{2/3} 1.25 \times 10^{-13} \text{ J}$$

$$= 2.1 \times 10^{24} \text{ m}^{-2}$$

$$\mathbf{N} / \mathbf{V} = 3.1 \times 10^{36} \text{ m}^{-3}.$$

This is a very high density. If all of the electrons in the Sun were concentrated at this density, they would occupy a volume of

$$\mathbf{V} = 10^{57} / 3.1 \times 10^{36} = 3 \times 10^{20} \text{ m}^3 = 3 \times 10^{11} \text{ km}^3.$$

The radius corresponding to this volume would be ~ 4000 km, somewhat smaller than the Earth (6400 km) and MUCH smaller than the Sun (700,000 km).

Neutron stars form from larger objects than the Sun, so the radius for the onset of the conversion reaction $e^- + p \rightarrow n + \nu$ is correspondingly higher. However, the radius of a stable neutron star once the conversion is complete, is much smaller than 4000 km, in the 10 - 15 km range.