APPENDIX A

REVIEW OF KINEMATICS

This appendix contains a brief review of kinematics and dynamics, approximately equivalent to the level of Grade 12 physics. Section A.I summarizes kinematics in Cartesian coordinates while Section A.II covers dynamics in Cartesian coordinates. Section A.III focuses on circular motion, material which may not be familiar to all students. Students do not need to understand all the applications given in Appendix A in order to understand the material in the main text.

A.I Kinematics in Cartesian Coordinates

To describe the motion of an object, we need a set of reference points against which to measure the position of the object as a function of time. Ignoring the many space-time subtleties involved in setting up such a reference system (dealt with in special relativity), in this section we define a set of three mutually perpendicular axes *X, Y* and *Z* by which the object's position is defined. The Cartesian coordinate *x, y* or *z* of the object is the distance along the *X, Y* or *Z* axis from the point at which the axes intersect (called the origin *O*) to the intersection point of the axis with a line drawn from the object perpendicular to the axis.
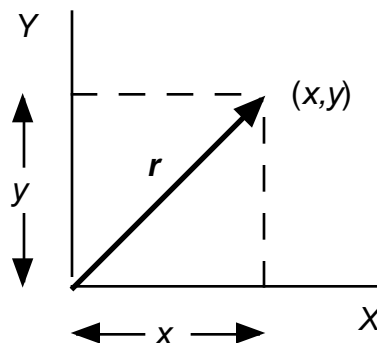

Fig. A.1. Cartesian coordinates of an object at position *r* in the *xy* plane.

The position of an object with coordinates (x, y, z) is labelled by a position vector $r(x, y, z)$, where the notation $r(x, y, z)$ just means that r is a function of the coordinates x, y and z.   Position vectors have the same general properties as other vectors. For example, two vectors a and b can be added together to form a third vector c. This operation is written as $c = a + b$. Vector addition is shown in Fig. A.2 for vectors in two dimensions.
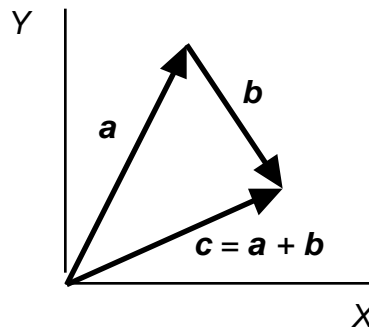

Fig. A.2. Addition of two vectors a and b to form a new vector c.

It is clear from Fig. A.2 that the Cartesian components of vector c are just the sum of the Cartesian components of vectors a and b. That is, if the components of a and b are (x_a, y_a, z_a) and (x_b, y_b, z_b) respectively, then the coordinates of c are $x_c = x_a + x_b$ etc. Subtraction of vectors $k = a - b$ is equivalent to the addition to a of the vector $-b$.   If vector b has components (x_b, y_b, z_b) then $-b$ has components ($-x_b$, $-y_b$, $-z_b$).   Hence vector k has components $x_k = x_a - x_b$ etc.

Let's apply this to our discussion of positions. The relative position of an object at $r_2(x_2, y_2, z_2)$ with respect to position $r_1(x_1, y_1, z_1)$ is defined as the *displacement vector r_{21}*:

$$r_{21}(x_{21}, y_{21}, z_{21})   r_2(x_2, y_2, z_2) - r_1(x_1, y_1, z_1). \tag{A.1}$$

Equation (A.1) is a vector equation in that it applies to each of the Cartesian components of the vector r_{21}. Cartesian coordinates have the simplicity that

$$x_{21} = x_2 - x_1 \qquad y_{21} = y_2 - y_1 \qquad z_{21} = z_2 - z_1. \tag{A.2}$$

If the object is moving with respect to the origin O, then $r(x, y, z)$ is a time-dependent quantity: $r(x(t), y(t), z(t))$. Suppose that the object is at

position $r_1(x_1, y_1, z_1)$ at time $t = t_1$ and position $r_2(x_2, y_2, z_2)$ at time $t = t_2$. We will assume below that $t_2 > t_1$. The magnitude of the displacement vector between these two positions is just the absolute magnitude of $r_{21}(x_{21}, y_{21}, z_{21})$ as determined by Eq. (A.1). We denote the magnitude of the displacement as

$$d = |r_{21}(x_{21}, y_{21}, z_{21})| = (x_{21}^2 + y_{21}^2 + z_{21}^2)^{1/2}. \qquad (A.3)$$

The displacement depends only on the endpoints of the path between positions 1 and 2, it does not depend on the length of the path taken between these endpoints. The length of the actual path followed between the endpoints is called the *distance*, *s*. It is clear that the positions of the endpoints uniquely define the displacement. However, we have to know the path in order to determine the distance. Note that if after some time the object ends up where it started from, its displacement is zero even if the distance it covers between the start and finish of its motion is not zero.

For moving objects, we are interested in the rate of change of position. But we have two ways of measuring this change of position: displacement and distance. The *average velocity* v_{av} is defined as the displacement (change in position) divided by the change in time:

$$v_{av} \quad r_{21}(x_{21}, y_{21}, z_{21}) \ / \ (t_2 - t_1). \qquad (A.4)$$

Note that this is a vector equation: it applies to each component *x*, *y*, and *z* separately. The *average speed* is defined as the distance divided by the change in time:

$$[average\ speed] \quad [distance] \ / \ (t_2 - t_1). \qquad (A.5)$$

Unlike Eq. (A.4), Eq. (A.5) is not a vector equation: it has no components. Speed has no direction, only a magnitude.

Because speed and velocity can change with time, just as position can, then the average speed or average velocity is not a very detailed description of the motion. A quantity which measures the motion at a particular time is the *instantaneous velocity*, *v*. We define it just as we did the average velocity, only now we let the difference between the two

times t_1 and t_2 become very small: $t = (t_2 - t_1)$ → 0. Then the instantaneous velocity is

$$v(t) → r_{21}(x_{21}, y_{21}, z_{21}) / (t_2 - t_1) \quad \text{as} \quad t → 0. \tag{A.6}$$

The *speed* is just the magnitude of the instantaneous velocity:

$$[speed] = |v|. \tag{A.7}$$

Finally, we complete our definition of elementary kinematical quantities by defining the acceleration. As with velocity, we can define the *average acceleration* a_{av} by

$$a_{av} → [v(t_2) - v(t_1)] / (t_2 - t_1) \tag{A.8}$$

and the *instantaneous acceleration* by

$$a → [v(t_2) - v(t_1)] / (t_2 - t_1) \quad \text{as} \quad t → 0. \tag{A.9}$$

Both of Eqs. (A.8) and (A.9) are vector equations.

The motion of an object at constant acceleration has a simple solution. At time $t = 0$, we assume that the particle is at position $r = 0$, and moves with (instantaneous) velocity v_i. At constant acceleration, then the average acceleration is equal to the instantaneous acceleration. From Eq. (A.9), we then have:

$$v_f = v_i + a\, t \tag{A.10}$$

The velocity is a function of time in this problem, as shown in Fig. A.3. To determine the position as a function of time, we must evaluate a sum over small time intervals t of the product of the instantaneous velocity with t. This procedure is the same as finding the area under the curve of velocity plotted as a function of time. The area is simple to calculate and can be written as a sum of two terms:
i) the area of a rectangle of base t and height v_i, which gives $v_i t$
ii) the area of a triangle of base t and height $(v_f - v_i)$, which gives $(v_f - v_i)t/2$.
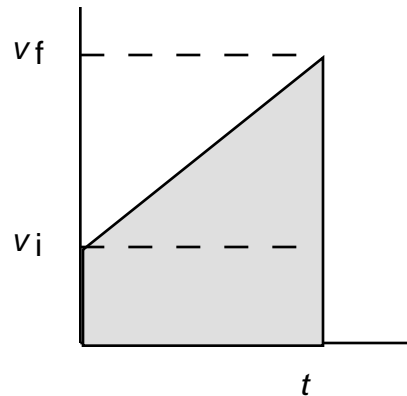
Fig. A.3. Plot of velocity vs. time for an object with constant acceleration in one dimension. The change in position of the object is equal to the cross-hatched area under the curve.

Adding the areas in (i) and (ii), the displacement is $d = v_i t + (v_f - v_i)t/2$, which can be rewritten using $v_f - v_i = at$ from Eq. (A.10) to give:

$$d = v_i t + 1/2\ at^2. \hspace{3cm} (A.11)$$

As implied by the boldface letters, Eq. (A.11) is a vector equation which applies to each of the components of d, v_i and a separately. For motion in one dimension, we can use Eq. (A.10) to express the time as $t = (v_f - v_i)/a$. This allows us to rewrite Eq. (A.11) as

$$d = (v_f^2 - v_i^2)\ /\ 2a. \hspace{3cm} (A.12)$$

A.II Dynamics in Cartesian Coordinates

The *kinematics* which we discuss in Section A.I is largely descriptive: it tells us how to describe the motion of a particle after the fact. The study of *dynamics* relates the object's motion to *forces* acting on it. The fundamental relationship is summarized in several physical laws formulated by Sir Isaac Newton in the late 1600's. Newton's First Law states:

A body at rest remains at rest and a body in motion continues in uniform motion unless acted upon by an external force.

This tells us what happens if there is *no* force present. If there is a force, then Newton's Second Law says that

A force acting on a body causes an acceleration which is in the direction of the force and has a magnitude which is inversely proportional to the mass of the body,

$$F = ma. \tag{A.13}$$

As we have written it, Newton's Second Law relates force to acceleration, which is in turn related to velocity and position by kinematics. In some situations, it is more appropriate to use an alternate quantity to velocity called momentum, p. As we define it, p is a vector quantity. In terms of p, Newton's Second Law reads

$$F = \ p \ / \ t \qquad \text{as } t \quad 0. \tag{A.14}$$

In other words, the force is equal to the rate of change of momentum. For small velocities and situations in which the force does not depend on the momentum, then

$$p = mv. \tag{A.15}$$

There are many situations in which we prefer to work with momentum rather than velocity, particularly when we are dealing with objects travelling at very high speeds.

Just for completeness sake, we should also mention Newton's Third Law, which has to do with the forces between bodies:

Whenever a body exerts a force on another body, the latter exerts a force of equal magnitude and opposite direction on the former.

An example of Newton's Third Law is the situation in which two skaters are standing motionless on an ice surface, each with their skate blades pointing towards the other skater. If one skater pushes on the other, then they move away from one another and they *both* move with respect to the ice.

A force acting on a body can change its velocity. If the body moves through a distance while the force is acting, then the force is said to do *work* on the body. If the force is constant and the body moves in a straight line, then the amount of work done is given by:

$$W = F \ x \tag{A.16}$$

where x is the change in the body's position in the direction of the force while the force is acting. Since constant force implies constant acceleration, we can use Eqs. (A.12) and (A.13) to substitute for the force and find

$$W = m \ [\ (v_f{}^2 - v_i{}^2) \ / \ 2 \quad x \ ] \quad x$$

$$= mv_f{}^2/2 - mv_i{}^2/2. \tag{A.17}$$

The work results in a change to the quantity which we call the kinetic energy:

$$K \quad mv^2/2 = p^2/2m. \tag{A.18}$$

In other words, the work done on the system increases its kinetic energy from $mv_i{}^2/2$ to $mv_f{}^2/2$.

A.III   Circular Motion

The kinematics of some simple situations is cumbersome to describe using Cartesian coordinates. As an illustration, consider the motion of an object moving at constant speed in a circular path of radius *R*. Since the speed is constant, it is tempting to say that the velocity is constant, but this is not true. While it is true that the *magnitude* of the velocity (which is just the speed) is *constant*, the *direction* of the velocity *changes* continuously. We show the effect in Fig. A.4 for motion in the *xy*-plane.

To describe uniform circular motion, we introduce the period *T* as the time taken for one complete orbit of the circular path. Then the speed is simply equal to the circumference of the circle divided by the period:

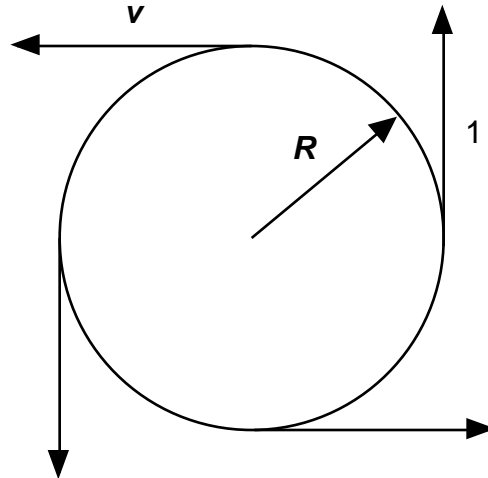$$|\mathbf{v}| = 2 \ R/T. \tag{A.19}$$

Fig. A.4. Motion of a particle moving in a circular path of radius *R*. The velocity of the particle is shown at several sample points.

Fig. A.5 shows how the direction of the velocity changes with time.For example, at time 1 in Fig. A.4, the position vector points to the right while the velocity vector points to the top of the page. On Fig. A.5, which is a graph of (v_x,v_y), the velocity at time 1 has components (0, *v*). Some people find the form of Fig. A.5 surprising: they expect to see both the tip and the tail of the velocity vector sitting on a circular path. However, what is shown in Fig. A.5 is the true time evolution of the velocity vector. It is really Fig. A.4 which is confusing, since sample values of the velocity vector have been superimposed on a position diagram.
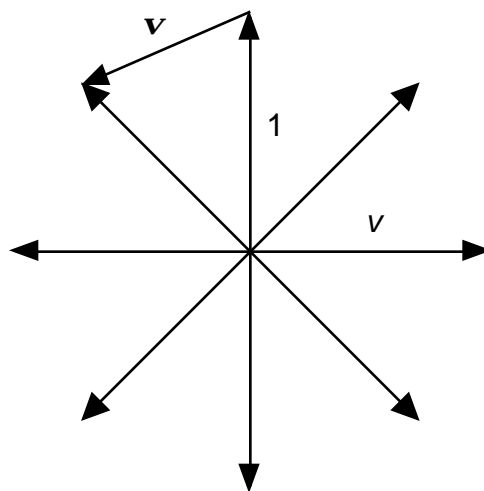

Fig. A.5. Change in velocity vector as a function of time for circular motion at constant speed. The total change in the velocity in the period *T* is equal to the perimeter of the circle of radius *v*.

From Fig. A.5, the total change in the velocity vector in time T is equal to the circumference of a circle of radius v. Hence, the magnitude of the acceleration is equal to

$$a = 2\ v/T \qquad\qquad\qquad\qquad\qquad (A.20a)$$
$$= v^2/R. \qquad\qquad\qquad\qquad\qquad (A.20b)$$

Eq. (A.20b) follows from solving Eq. (A.19) for T and substituting into Eq. (A.20a). The direction of the acceleration vector changes in time. The velocity vector is always perpendicular to the position vector R, as shown on Fig. A.4. The *change* in the velocity vector v is perpendicular to the velocity vector, as can be seen from Fig. A.5. Hence, the acceleration vector, which is proportional to the *change* in the velocity vector, must lie along the same axis that R does. Careful inspection of Figs. A.4 and A.5 shows that v points towards the centre of the circle at all times. Hence, the acceleration vector must point towards the centre of the circle, and we call it the *centripetal* (or *centre seeking*) *acceleration*.

We have now found the acceleration in terms of the speed and radius. But rather than complete the kinematical description of circular motion using Cartesian coordinates, which lead us into sine and cosine functions, we will instead develop a set of angular coordinates to describe the motion. Let's return to Fig. A.4 for a moment. Suppose that the initial position of the object is on the *x*-axis at time *t*=0. We define an angle ϕ as the angle between the position vector R and the *x*-axis. Then the arc length *s*, or distance which the object has traveled, is

$$s = \phi R. \qquad\qquad\qquad\qquad\qquad (A.21)$$

The speed, which is the rate of change of *s*, is then

$$[speed] = R\ (\phi_2 - \phi_1)\ /\ (t_2 - t_1). \qquad\qquad\qquad\qquad\qquad (A.22)$$

Since R is a constant in this situation, then it is the rate of change of ϕ which carries the information about the motion. We define the angular speed to be

$$\omega_{av} \quad [\textit{average angular speed}]$$
$$(\phi_2 - \phi_1)\ /\ (t_2 - t_1). \qquad\qquad\qquad\qquad\qquad (A.23)$$

We can also define an average angular acceleration α_{av} as the rate of change of the angular speed:

α_{av} [*average angular acceleration*]

$$(\omega_2 - \omega_1) / (t_2 - t_1). \qquad\qquad (A.24)$$

The quantities ϕ, ω and α are angular variables related to the corresponding Cartesian variables s, v and a_{tan} by

$$s = \phi R \qquad\qquad v = \omega R \qquad\qquad a_{tan} = \alpha R, \qquad\qquad (A.25)$$

where a_{tan} is the component of the linear acceleration that is *tangential* to the circle. For the situation we have just solved, uniform circular motion, ω and a_{tan} are constant, since v is constant, and

$$s = \omega R t \qquad \omega = v/R \qquad \alpha = 0. \text{ (uniform circular motion)} \qquad (A.26)$$

There are other angular variables related to the force and momentum. The kinetic energy for circular motion is

$$K = mv^2/2 = m\,(\omega R)^2/2 = (mR^2)\omega^2/2 = I\omega^2/2 \qquad\qquad (A.27)$$

where I is the moment of inertia, defined for a single particle as mR^2. The *angular momentum L* has the same relationship to the moment of inertia and the angular speed as the momentum does to the mass and speed:

$$L = I\omega. \qquad\qquad (A.28)$$

The angular equivalent of force is the torque, τ. Torque enters into Newton's Second Law as

$$\tau = I\alpha. \qquad\qquad (A.29)$$

Clearly, the formalism which we have developed for Cartesian coordinates has an equivalence in angular coordinates. Further, there are vector analogues to velocity and acceleration as well: vector angular velocity, ω, vector angular acceleration, α, and vector angular momentum, L. However,

to deal with these quantities properly here would take us too far from our purpose, and so we refer the interested reader to any introductory text on mechanics for more information.

Further Reading

P. A. Tipler, *Physics for Scientists and Engineers* (Worth, New York, ed. 3, 1991), Chaps. 1 - 6, 8.

H. D. Young, *University Physics* (Addison-Wesley, Reading, ed. 8, 1992), Chaps. 1 - 6, 8 - 9.

R. Resnick, D. Halliday and K. S. Krane, *Physics* (Wiley, New York, ed. 4, 1992), Chaps. 1 - 7, 11.