CHAPTER 4

BOUND SYSTEMS

4.A Binding Energy

The mass-energy equation (3.2) is valid for a single point-like elementary particle or a galaxy: the equation deals with the mass of a system, its overall linear momentum and its energy. While we introduced the equation in order to describe the energy of a single "particle", we now wish to apply it to composite systems. Consider a binary system *A* composed of two components denoted by *B* and *C*. For example, *B* might represent the Earth and *C* the Moon, with the binary system *A* being the combined Earth-Moon pair. Alternatively, *B* might be an electron and *C* a proton, with the combined pair *A* being a hydrogen atom. What is the mass of *A* compared to the masses of *B* and *C*? Based on our everyday experience, we are tempted to say that the mass of *A* is equal to the sum of the masses of *B* and *C*. While there are situations where this is true, for bound states it is not true.

The mass of a bound state is *less* than the sum of the masses of its components:

$$m_A < m_B + m_C. \qquad (4.1)$$

The reason why we don't notice this difference in our everyday world is that the difference in mass is so small that it may not even be measurable with present-day technology. But on the subatomic scale, the difference is measurable. The implication of (4.1) taken with (3.2) is that the mass energy of a bound state is less than the sum of the mass energies of its components. This energy difference is the *binding energy*, *B.E.*, of the system:

$$B.E. = m_B c^2 + m_C c^2 - m_A c^2. \qquad (4.2)$$

An aside on notation: in many texts, the binding energy is simply represented by *B*. However, to avoid confusion with baryon number, we explicitly denote the binding energy by *B.E.*

The binding energy can be viewed as the energy required to separate the components of a system so that each component has no kinetic energy and is infinitely far away from any other component. Equivalently, it is the energy that can be released from a system when its isolated components are brought together to form a bound state. The sign convention in Eq. (4.2) is arranged so that *B.E.* is positive for a bound system. In fact, (4.2) can be easily generalized to a system with any number of components by:

$$B.E. = \sum_i m_i c^2 - Mc^2. \tag{4.3}$$

where M is the mass of the bound system and the summation is over all i components.

Example 4.1: *The mass of a nucleus containing only a proton and a neutron is about 0.1% less than the sum of the individual proton and neutron masses. Approximately what is the binding energy of the pair?*

The mass difference is approximately 0.1% of $m_p + m_n$, the sum of the proton and neutron masses. That is

$$\Delta m = m_p + m_n - m_{p+n}$$

$$\approx (0.1 \times 10^{-2}) \times (m_p + m_n)$$

$$\approx 10^{-3} \times (1.673 + 1.675) \times 10^{-27} \text{ kg}$$

$$\approx 3.3 \times 10^{-30} \text{ kg}.$$

The binding energy corresponding to this mass difference is

$$B.E. = \Delta mc^2 = 3.3 \times 10^{-30} \times (3.0 \times 10^8)^2 = 3 \times 10^{-13} \text{ J}.$$

As we have defined it, the binding energy equation compares masses of particles or systems at rest: the components at infinite separation are at rest, as is the bound state. No attention need be paid to the "internal" structure or motion of the components of the bound system. In other words, in calculating the binding energy of a hydrogen atom only the mass of the hydrogen atom at rest is compared with the masses of the isolated components of the hydrogen atom (the proton and the electron) also at rest, and it does not matter whether the proton and electron are moving with respect to each other within the atom.

The term *binding energy* describes how much energy is required to break up a system into specified components. For example, the nuclear binding energy is how much energy it takes to break up a nucleus into isolated protons and neutrons. We use the words atomic binding energy to mean the energy required to break up an atom into isolated electrons and the (bound) nucleus. Clearly, the energy required to break up an atom into isolated electrons, protons and neutrons must be the sum of the atomic binding energy plus the nuclear binding energy.

4.B Binding Energy Systematics

The smallest building blocks of matter that can be isolated are the elementary particles such as protons, neutrons and electrons, that make up atoms. Although there is very strong evidence that hadrons, such as protons and neutrons, are made from quarks and gluons, these constituents of hadrons have not been isolated. Let’s start with the elementary particles as building blocks and find the binding energy scales of the atoms, solids, planets and stars that they can form.

Nuclei

According to Eq. (4.3), the binding energy of a nucleus is given by

$$B.E. = Zm_{\mathrm{p}}c^2 + Nm_{\mathrm{n}}c^2 - m(A, Z)c^2 \qquad (4.4)$$

where $m(A, Z)$ is the mass of a nucleus with Z protons, N neutrons and mass number A [see also Eq. (1.4)]. The binding energy varies with A and Z, and only a limited set of (A, Z) combinations correspond to stable nuclei. Most of the nuclei in our everyday elements have Z approximately equal to

N and these nuclei are among the deepest bound or most stable nuclei. Except for the very lightest and very heaviest of the nuclei, the binding energy of the most deeply bound nucleus with a given mass number *A* is approximately equal to

$$B.E. \approx 8A \quad \text{(in MeV)}. \tag{4.5}$$

Eq. (4.5) shows that the binding energy for each of the nucleons in the nucleus is roughly constant at 8 MeV. This should be compared to the mass energy for the nucleons (protons and neutrons) of 939 MeV, according to Table C.6. In other words, for the most deeply bound nuclei, the binding energy is about 1% of the mass energy.

Atoms and molecules

All stable nuclei contain at least one proton, and hence all stable nuclei are positively charged. In an atom, the positively charged nucleus is surrounded by negatively charged electrons, attracted to the nucleus with a force given by Coulomb's Law, Eq. (2.2). Taking the simplest atom as an example, the neutral hydrogen atom ($p + e^-$) has a binding energy of 13.6 eV in its most deeply bound arrangement. This binding energy is very small compared to the proton and electron mass energies of 938 and 0.5 MeV respectively. For hydrogen atoms, the atomic binding energy is therefore of the order of 10^{-8} of the atomic mass energy. The binding of atoms into molcules is even weaker than the binding of electrons to nuclei in atoms. For example, two hydrogen atoms can join to form a *diatomic molecule* H_2. The molecular binding energy of the two atoms in forming the molecule is 4.75 eV, compared to 2 x 13.6 = 27.2 eV for the binding energy of the electrons and protons to form the individual atoms.

The reason for the relatively small binding energy lies in the interaction which governs the binding. The strong interaction is responsible for the nuclear bound state, whereas the (weaker) electromagnetic interaction is responsible for atomic and molecular bound states. The chemical energy scale is set by the electromagnetic interaction, and one can see that there is about a factor of a million, give or take a factor of ten, between the nuclear and chemical energy scales. Put another way, the Sun would not still be shining had it been made out of coal and derived its energy by burning (chemically) in an oxygen-rich environment. The Sun's energy source is nuclear, not chemical, in origin.

Solids, liquids

Molecules can interact with each other to form small clusters, or to form bound systems of much larger sizes. The binding energies of liquids and solids are available from physical chemistry studies. For example, when water is converted from a liquid to a gas at 100 ^oC (and 1 atmosphere pressure), 0.42 eV must be added to each water molecule to overcome its binding with other water molecules in the liquid state. This example shows us that the binding of molecules into liquids involves energy scales that are weaker, but not dramatically so, than the energy scales of atomic binding. Again, this should come as no surprise because the interaction governing the binding of molecules into liquids *etc.* is electromagnetic in origin.

Planets, stars

Consider a solid sphere of matter with mass M and radius R. We wish to calculate the contribution of gravity to the binding of the sphere. In Eq. (2.1), the gravitational force between two objects of mass m_1 and m_2 separated by a distance r is

$$F = Gm_1m_2/r^2. \qquad (4.6)$$

Using integral calculus, it is not difficult to calculate the gravitational binding energy of a sphere; it's just a matter of summing Eq. (4.6) over all of the small bits of mass in the sphere. The energy that is released when matter is formed into a sphere of uniform density (and no internal motion of its constituents) is found to be

$$B.E. = (3/5)\ GM^2/R. \qquad (4.7)$$

Example 4.2 shows once again that gravity is indeed a weak force. A 1 g drop of water has a gravitational binding energy of about 10^{-18} eV per molecule, compared to about 1/2 eV per molecule for the binding of individual water molecules in the droplet. That is, the drop of water is held together by electromagnetic interactions among its molecules, not by gravity. Where is gravity important, then? The answer lies in the scaling behaviour with mass of Eq. (4.7). The numerator clearly scales like M^2. The radius of an object with constant density scales like $M^{1/3}$, meaning that the binding energy as a whole scales like $M^{5/3}$. So, gravity dominates at large masses since the gravitational binding energy grows faster than M^1.

Example 4.2: *Calculate the gravitational binding energy of a spherical droplet of water of mass 1 g. Assume that the water has a density of 1 g/cm^3.*

First, we need to find the radius of the drop. The volume of a sphere of radius R is (4/3) πR^3. The sphere's radius can be found by manipulating the relationship [*mass*] = [*density*] x [*volume*] to yield

$$R = (3/4\pi)^{1/3} \times (1[gram] / 1[gram/cm^3])^{1/3} = 0.62 \text{ cm.}$$

Converting the units to MKSA, then a substitution into Eq. (4.7) yields

$$B.E. = 3/5 \times [6.67 \times 10^{-11}] \times [10^{-3}]^2 / [6.2 \times 10^{-3}]$$

$$= 6.5 \times 10^{-15} \text{ J.}$$

Converting units, the gravitational binding energy is about 40,000 eV, spread over 3 x 10^{22} molecules, or about 10^{-18} eV per molecule.

4.C Energetics of Reactions and Decays

Particles interact, and their interactions can cause changes in their momenta, energies and other characteristics. A *reaction* involves the interaction of two or more particles, usually in motion with respect to one another. A *decay* usually refers to the change that a single system undergoes, often resulting in the emission of more particles (for example, photons). Consider a two-body reaction in which particle A interacts with particle B to change into particles C and D

$$A + B \rightarrow C + D. \qquad (4.8)$$

Although we use the word "particle", A, B, C and D do not have to represent elementary particles: they can be composite systems, galaxies, toothbrushes *etc*. Further, not all of the particle characteristics need to change during the reaction: a red toothbrush (A) bouncing off a chocolate doughnut (B) may emerge as a spinning red toothbrush with chocolate on it (C), but we would still say that it is a red toothbrush. So too with elementary particles: two protons may scatter from each other and exchange energy and momentum instead of chocolate, but they still emerge as protons.

Reactions and decays are subject to *conservation laws*: there exist *conserved* quantities that do not change during a reaction. Total energy E and momentum $\boldsymbol{p}$ are conserved in reactions and decays. Mathematically, this is written for reaction (4.8) as:

energy $$E_A + E_B = E_C + E_D \tag{4.9}$$

momentum $$\boldsymbol{p}_A + \boldsymbol{p}_B = \boldsymbol{p}_C + \boldsymbol{p}_D. \tag{4.10}$$

The subscripts on the symbols $\boldsymbol{p}$ and E refer to the particle labels in Eq. (4.8). For example, Eq. (4.9) reads "the energy of particle A plus the energy of particle B (before the reaction) is equal to the energy of particle C plus the energy of particle D (after the reaction)". Eq. (4.10) for the conservation of momentum is actually three equations in one: it applies separately to each component of the momentum (p_x, p_y, p_z) separately.

A conservation law does *not* mean that the conserved characteristic (e.g., the energy) of each individual particle remains the same throughout the interaction process. Rather, it is the conserved characteristic of the system as a whole, summed over all of its components, that remains the same. In symbols, E_A and E_B individually may change, but $E_A + E_B$ is constant. Similar equations to (4.9) and (4.10) can be written for decay processes, $A \rightarrow B + C$, but there is only one term (E_A or $\boldsymbol{p}_A$) on the left hand side of the equations because the initial state consists of a single object.

Example 4.3: *An electron (e^-) and a positron (e^+) annihilate head on to produce two gamma rays. The reactants each carry 1.00 MeV of kinetic energy into the reaction. What are the kinetic energies of the gamma rays? The electron has a mass energy of 0.51 MeV.*

This problem illustrates conservation of energy and momentum. The reaction is written symbolically as

$$e^+ + e^- \rightarrow \gamma + \gamma.$$

The initial energy of each of the particles is

$$E = K + m_e c^2 = 1.00 + 0.51 = 1.51 \text{ MeV}$$

so the total energy before and after the reaction takes place is 2 x 1.51 = 3.02 MeV. After the reaction, this energy is split between the two gamma rays. But how is it split?

The magnitude of each electron's momentum is the same because they each have the same kinetic energy. But the momenta have opposite signs since the electrons are heading towards each other. The sum of the initial momenta, taking into account the signs, must be zero. After the reaction, the total momenta of the gamma rays still must be zero, according to Eq. (4.10). Hence, the gammas must also have equal and opposite momenta. For a massless particle like a gamma ray, $E = pc$ so that the gammas must have equal energies if they have equal momenta. Therefore, the energy of each gamma must be 3.02 / 2 = 1.51 MeV.

A particle or system that decays spontaneously may liberate kinetic energy. Consider the decay $A \rightarrow B + C$ from an observational frame in which particle A is at rest. Then the initial energy of the system is just $m_A c^2$. The total energy of particles B and C after the decay is equal to the sum of the mass energies, $m_B c^2 + m_C c^2$, plus the kinetic energies of particles B and C. If the initial mass energy $m_A c^2$ is greater than the final mass energy $m_B c^2 + m_C c^2$ then the decay can occur spontaneously, liberating kinetic energy. If $m_A c^2$ is less than $m_B c^2 + m_C c^2$, then the reaction is forbidden by conservation of energy. The initial mass energy less the final mass energy is called the Q-value of the reaction:

$$Q\text{-value} = m_A c^2 - (m_B c^2 + m_C c^2). \tag{4.11}$$

In most texts, the *Q*-value is simply abbreviated as *Q*, but here we write it out in full to avoid confusion with the definition of *Q* for the electric charge. The energy that particles *B* and *C* can carry off as kinetic energy is equal to the *Q*-value if particle *A* is at rest. Clearly, a decay is forbidden if the *Q*-value is negative.

Example 4.4: *Calculate the Q-value for the decay of a free neutron*

$n \rightarrow p + e^- + \text{anti-}\nu_e$

where the last particle in the equation is an electron anti-neutrino with zero mass.

The difference in masses Δm for the reaction is

$\Delta m = m_n - m_p - m_e - m_\nu$

$= 1.6750 \times 10^{-27} - 1.6726 \times 10^{-27} - 0.00091 \times 10^{-27} - 0$

$= 1.5 \times 10^{-30}$ kg.

Hence, the *Q*-value is

Q-value $= 1.5 \times 10^{-30}$ [kg] x $(3.0 \times 10^8$ [m/s]$)^2$

$= 1.35 \times 10^{-13}$ J = 0.84 MeV.

Note that this number is much less than the typical binding energy per nucleon in a nucleus, which is about 8 MeV.

The *Q*-value can be generalized to include reactions as well:

$$Q\text{-value} = \Sigma_{\text{initial}}\, mc^2 - \Sigma_{\text{final}}\, mc^2, \tag{4.12}$$

where the sums are over the initial reactants and the final products of the reaction. The *Q*-value gives the kinetic energy that can be released in a reaction or decay. If the initial particles are moving with respect to the observational frame, then the kinetic energies of the reaction products may be higher than what is given by the Q-value. Indeed, relative motion of the initial particles may allow a reaction to proceed which otherwise would be forbidden by a negative Q-value. However, adding kinetic energy to a single particle will not influence whether it can decay spontaneously: a decay with negative Q-value is forbidden.

Summary

When particles form a bound state, the total mass M of the bound state is less than the sum of its component masses in isolation. The binding energy of the bound state is equal to this difference in masses times c^2, as in Eq. (4.3):

$$B.E. = \sum_i m_i c^2 - Mc^2.$$

The binding energy for the most stable nucleus with a given mass number A is governed approximately by Eq. (4.5),

$$B.E. \approx 8A \quad \text{(in MeV)}.$$

Typical values for the binding energy of atoms, molecules and liquids are 10, 1 and 0.1 eV per particle, respectively. The gravitational binding energy of a uniform density sphere of mass M and radius R (and no internal motion of its constituents) is given by Eq. (4.7)

$$B.E. = (3/5)\, GM^2/R.$$

Reactions such as $A + B \rightarrow C + D$ are subject to conservation laws of the form [Eqs. (4.9) and (4.10)]

$$\text{energy} \qquad E_A + E_B = E_C + E_D$$

$$\text{momentum} \qquad \mathbf{p}_A + \mathbf{p}_B = \mathbf{p}_C + \mathbf{p}_D.$$

The Q-value of a reaction or decay is the difference between the mass energies of the initial reactants and the final products, or

$$Q\text{-value} = \Sigma_{\text{initial}}\, mc^2 - \Sigma_{\text{final}}\, mc^2$$

according to Eq. (4.12). A decay is forbidden if its *Q*-value is negative, whereas reactions are allowed even for negative *Q*-values, so long as the initial reactants have sufficient kinetic energy to overcome the deficit in mass energy between the reactants and the products.

Further Reading

P. A. Tipler, *Physics for Scientists and Engineers* (Worth, New York, ed. 3, 1991), Chaps. 10, 18.

S. Weinberg, *The Discovery of Subatomic Particles* (Scientific American, New York, 1983), Chap. 4.

Problems

1. A deuteron (proton + neutron bound state) has a binding energy of 2.2 MeV (1 MeV = 10^6 eV). What is the mass energy of the deuteron?

2. An electron and a proton can bind together to form a hydrogen atom, with a binding energy of 13.6 eV. What is the mass of a hydrogen atom?

3. The binding energy due to gravity of two objects with masses m_1 and m_2 moving in a stable orbit separated by a distance R is $Gm_1m_2/2R$. (a) Find the binding energy of the Earth/Moon system. (b) Compare this binding energy to the sum of their mass energies.

*4. (a) Use Eq. (A.16) to show that the work required to move an object of mass m through a vertical height difference h near the surface of the Earth is mgh, where g is the acceleration due to gravity and where the object is at rest before and after the move.
(b) Which system has more mass and by how much: (i) the Earth plus a 1 kg box of cookies at rest on the surface of the Earth, or (ii) the Earth plus a 1 kg box of cookies at rest 1 m above the surface of the Earth?

5. A star radiates light at a rate of 4 x 10^{26} J/s. By approximately how much does a star's mass change per second?

6. It has been proposed that at very high densities or temperatures, quarks and gluons become unbound to form the so-called quark/gluon plasma. If it takes 300 MeV in energy to liberate each quark, how much energy (in J) is required to turn the Sun into a quark star?

7. Calculate the total decrease in the Earth's mass associated with
(a) gravitational binding, using Eq. (4.7)
(b) nuclear binding, using Eq. (4.5).

8. A medium-size city consumes 10^9 J of electricity per second. Suppose that this energy demand could be met with energy released in the *fusion* of hydrogen nuclei to form helium. If 28 MeV of binding energy is released when each helium nucleus is formed, then what mass of helium would be produced in a year in generating 10^9 J/s?

9. The Sun is about 75% by weight hydrogen and 25% by weight helium. Calculate the change in the mass of the Sun if all of the helium nuclei were broken up into their constituent nucleons. (Take the binding energy of helium nuclei to be 28 MeV and assume that a helium atom has approximately the same mass as 4 protons).

10. The work required to place a total charge Q into a spherical volume of radius R is, in analogy with Eq. (4.7)

$$W = (3/5)\, kQ^2/R. \qquad (4.13)$$

Find the (electromagnetic) work required to bring the protons together in carbon (A=12) and iron (A=56) nuclei. Use Eq. (1.5) for the nuclear radii.

11. Use Eq. (4.13) to determine at what mass number A the Coulomb energy of repulsion in the nucleus exceeds 8 MeV per nucleon. Assume that $A = 2Z$. Use Eq. (1.5) for the nuclear radius.

12. Use Eq. (4.13) to determine the work required to bring
(a) 20 electrons together in a spherical volume of radius 2 Å,
(b) 20 protons together in a spherical volume of radius 4 fm.
On a per particle basis, how do these electromagnetic repulsion energies compare with typical atomic or nuclear binding energies?

*13. What is the change in mass of 5 kg of lead when it melts at 330 ^{o}C with no change in temperature? The latent heat of fusion of lead is 5 kcal/kg.

*14. What is the change in the mass of 1 mole of ice at 0 ^{o}C when it changes to water at 0 ^{o}C? The latent heat of fusion of ice is 79.7 kcal/kg.

15. Use Eq. (4.7) for the gravitational binding energy of a solid sphere to find the binding energy of the Moon due to its own gravity.

16. Use Eq. (4.7) for the gravitational binding energy of a solid sphere to find the change in mass of Jupiter when it formed from a very dilute gas of hydrogen.

17. Which of the following decays are forbidden by conservation of energy?

(a) $n \to p + \pi^-$

(b) $K \to \pi + \pi + \pi$

(c) anti-p $\to$ anti-n + e^- + anti-ν_e.

18. Find the *Q*-values of the following reactions and decays. Determine if the processes can occur spontaneously:

(a) $e^+ + e^- \to \gamma + \gamma$

(b) $e^+ + e^- \to$ p + anti-p

(c) $K \to \pi + e^+ + \nu_e$

(d) $\pi \to e + \nu_e$

(e) $\pi^o \to \mu^+ + \mu^-$.

19. A neutral pion can decay into a neutrino-antineutrino pair, $\pi^o \to \nu$ + anti-ν, although the process is uncommon.

(a) Calculate the *Q*-value for this decay, quoting your answer in MeV.

(b) Assuming that the initial pion is at rest, find the momenta of the neutrinos.

(c) Determine the de Broglie wavelength of the neutrinos, quoting your answer in fm.

20. Suppose that the reaction powering the Sun is the *molecular* reaction $H + H \to H_2$; that is, two hydrogen atoms join to form one hydrogen molecule, giving off 4.75 eV in the process.

(a) Assuming that the Sun is pure hydrogen, estimate the total energy available through this reaction. Express your answer in Joules.

(b) If the Sun radiates energy at the rate of 3.9×10^{26} J/s, estimate how long it could shine on the basis of this molecular reaction alone.