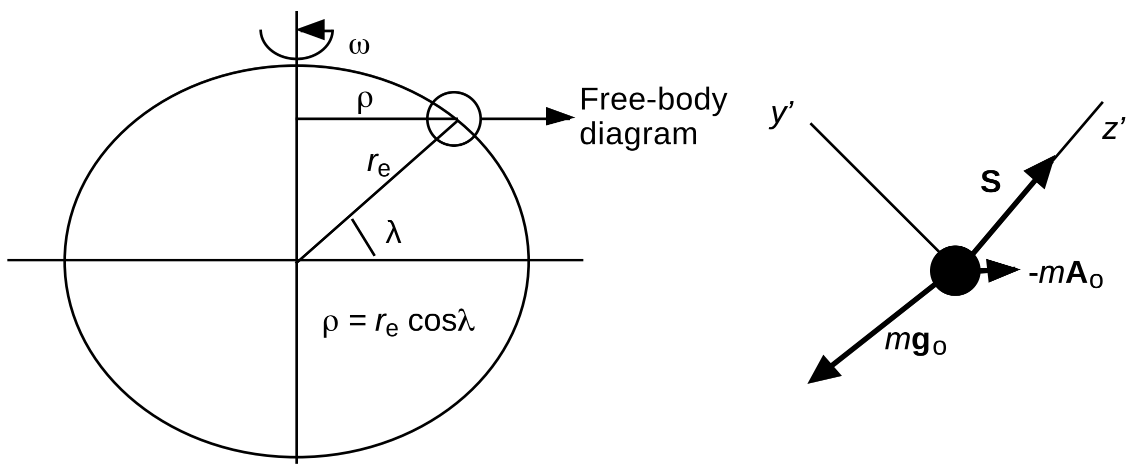
Lecture 12 - The Earth's rotation

Text: Fowles and Cassiday, Chap. 5
Demo: basketball for the earth, cube for Cartesian frame

A coordinate system sitting with fixed orientation on the Earth's surface (*i.e.*, having the*y*-axis pointing towards the North Pole, the *x*-axis pointing east along a line of latitude and the *z*-axis perpendicular to the Earth's surface) is both a rotating and an accelerating reference frame. Hence, fictitious forces are required in this frame to account for an object's motion. The Earth rotates counter-clockwise as seen at North Pole, so ω is up.

<u>The Plumb Line</u>

Our first example is a simple plumb line: a point mass hung from a string. The coordinate system is ilustrated below



λ = latitude (angle)
ρ = distance of plumb line from axis of rotation
r_e = distance of plumb line from the centre of the Earth

We've drawn the Earth as if it has a slight bulge at the equator. Although we've drawn the tension in the plumb line **S** as if it points along *z'*, in fact we don't know that is the case yet (*we use S instead of T, which is reserved for period*).

This is a particularly simple example because the plumb bob is stationary in the rotating frame. How do we describe the bob?

(i) it is stationary in the rotating frame: $\mathbf{a}' = 0$ and $\mathbf{v}' = 0$
(ii) it sits at the origin of the frame so $\mathbf{r}' = 0$
(iii) the Earth's rotational speed is constant, so $d\omega / dt = 0$

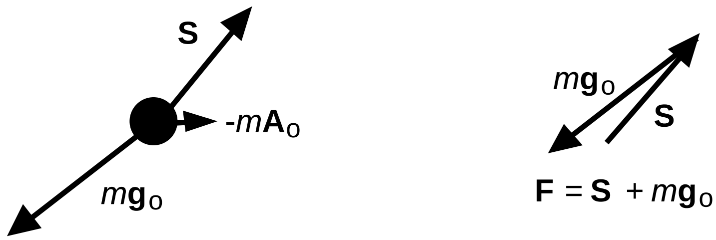
Thus, of the expression

$$m\mathbf{a}' = m\mathbf{a} - 2m\omega\mathbf{x}\mathbf{v}' - m\,(d\omega / dt)\mathbf{x}\mathbf{r}' - m\omega\mathbf{x}(\omega\mathbf{x}\mathbf{r}') - m\mathbf{A}_o$$

we are left with

$0 = m\mathbf{a} - m\mathbf{A}_o$ or just $\mathbf{F} = m\mathbf{A}_o$

Now the physical forces **F** acting on the bob are the true gravitational force $m\mathbf{g}_o$ toward the centre of the Earth and the tension **S** in the string.



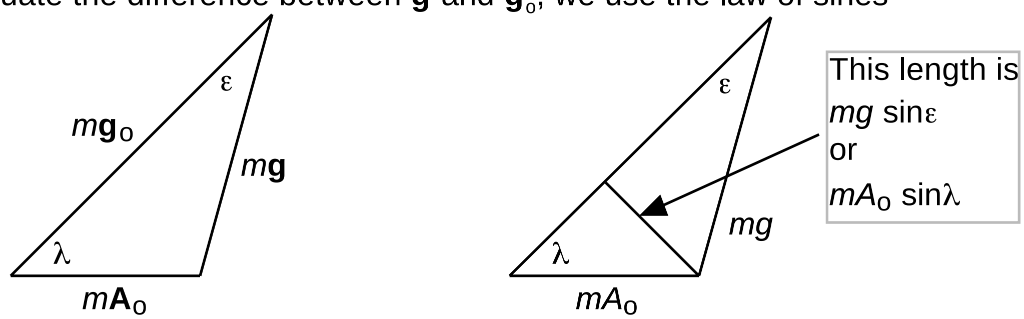
The tension balances the locally measured acceleration due to gravity **g**, which is different from the true acceleration $\mathbf{g}_o$. That is

$\mathbf{S} = -m\mathbf{g}$

=> $\mathbf{F} = -m\mathbf{g} + m\mathbf{g}_o = m\mathbf{A}_o$ (vector equation)

or $\mathbf{g} = \mathbf{g}_o - \mathbf{A}_o$

To evaluate the difference between **g** and $\mathbf{g}_o$, we use the law of sines



$\sin\varepsilon / mA_o = \sin\lambda / mg$

$\mathbf{A}_o$ is the centri*petal* acceleration (and $-\mathbf{A}_o$ is the centri*fugal* acceleration) given by

$A_o = \omega^2\rho = \omega^2 r_e \cos\lambda$

--> $\sin\varepsilon = m\omega^2 r_e \sin\lambda \cos\lambda / mg$

$= \omega^2 r_e \sin 2\lambda / 2g$

As expected, ε vanishes at the equator (λ = 0) and at poles (2λ = 180 degrees). The largest value of ε is at λ = 45 degrees:

$\omega = 2\pi / (24 \times 3600) = 0.000073 = 7.3 \times 10^{-5}$ radians/sec

$r_e = 6400$ km $= 6.4 \times 10^6$ m

$\sin\varepsilon \sim \varepsilon = (7.3 \times 10^{-5})^2 \bullet 6.4 \times 10^6 \bullet 1 / (2 \times 9.8) = 0.0017$ radians

or ε ~ 0.0017 x 57.3 = 0.1 of a degree.

Lastly, because the rotational motion affects both the surface of the Earth as well as the bob, then **S** remains perpendicular to the Earth's surface. We use this fact in the next example, and always subsume $\mathbf{A}_o$ with $\mathbf{g}_o$ to allow us to work with the local acceleration $\mathbf{g}$. At the equator, $A_o = (7.3 \times 10^{-5})^2 \cdot 6.4 \times 10^6 = 0.034$ m/s^2.

Foucault Pendulum

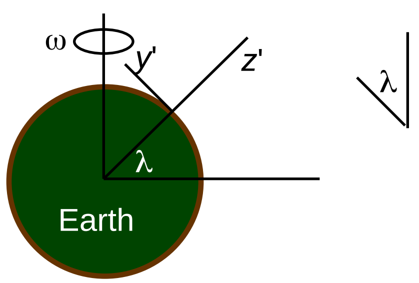
Now we take the plumb bob from the previous example and allow it to swing back and forth. The Foucault pendulum is simply a mass on a string, but the string is not restricted to move in a particular plane.

The forces acting on the pendulum bob are

$$m\,(d^2\mathbf{r}' / dt^2) = m\mathbf{g} + \mathbf{S} - 2m\omega\mathbf{x}\mathbf{v}',$$

where we have rolled $\mathbf{g}_o - \mathbf{A}_o$ into the local acceleration $\mathbf{g}$. But because the tension $\mathbf{S}$ changes magnitude and direction as the bob swings, it is not always true that $\mathbf{S} = -m\mathbf{g}$. The local centripetal acceleration term $\omega\mathbf{x}(\omega\mathbf{x}\mathbf{r}')$ has been dropped in favour of the Coriolis force, which is much more important for this problem [The local force $\omega\mathbf{x}(\omega\mathbf{x}\mathbf{r}')$ depends upon $\mathbf{r}'$ which is with respect to the $x'y'z'$ origin on the surface of the Earth, whereas $\mathbf{A}_o$ depends on $r_e \ggg r'$].

Choose x', y' to form a plane tangent to the Earth's surface, and choose z' to therefore lie along $\mathbf{g}$:



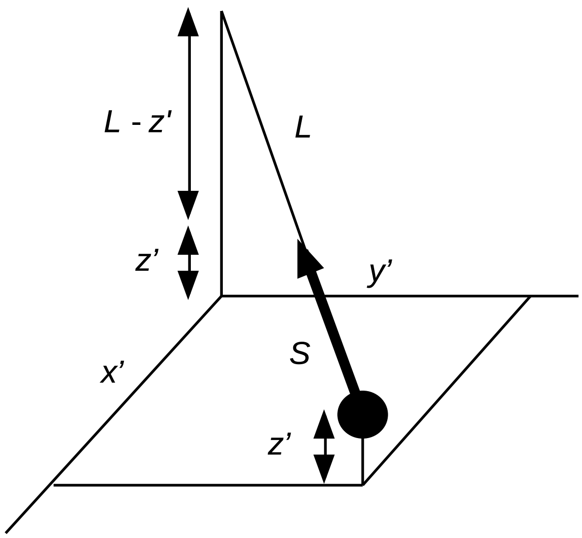
$\omega_{x'} = 0$
$\omega_{y'} = \omega \cos\lambda$
$\omega_{z'} = \omega \sin\lambda$

From $\mathbf{v}' = v_{x'}\mathbf{i}' + v_{y'}\mathbf{j}' + v_{z'}\mathbf{k}'$, the Cartesian components of $\omega\mathbf{x}\mathbf{v}'$ are

$$\omega\mathbf{x}\mathbf{v}' = [\omega\cos\lambda\,(dz'/dt) - \omega\sin\lambda\,(dy'/dt),\quad \omega\sin\lambda\,(dx'/dt),\quad -\omega\cos\lambda\,(dx'/dt)]$$

Now the components of the tension $\mathbf{S}$ can be written in terms of the length of the string and the coordinates of the bob: comparing S_x / S with x' / L, for example, gives

$$S_{x'} = -(x'/L)S.$$



$S_{x'} = -(x' / L)S$

$S_{y'} = -(y' / L)S$

$S_{z'} = +[(L - z') / L]S$

Note that the - signs are required to obtain the correct orientations.

Returning to our equation for **a**', we can write out the x' and y' components as

$$m\,(d^2x' / dt^2) = -(x' S / L) - 2m\omega\,[(dz' / dt)\cos\lambda - (dy' / dt)\sin\lambda]$$

$$m\,(d^2y' / dt^2) = -(y' S / L) - 2m\omega\,(dx' / dt)\sin\lambda$$

We make two approximations in the small angle situation, where the motion is nearly horizontal:

- $dz' / dt \sim 0$ corresponding to no vertical motion
- $S = mg$ (in magnitude) since the pendulum is almost vertical

=>

$$d^2x' / dt^2 = -(g / L)\,x' + 2\Omega\,(dy' / dt)$$

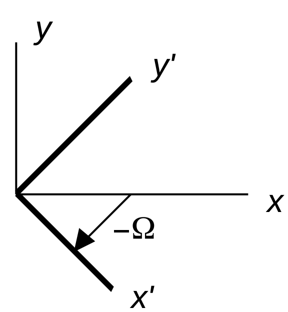
$$d^2y' / dt^2 = -(g / L)\,y' - 2\Omega\,(dx' / dt)$$

where $\Omega = \omega\sin\lambda$ is the component of ω in the z' direction, *i.e.*, it's the local vertical component of ω.

Somewhat like the *q***vxB** problem of charged particles in a magnetic field, the equations of motion are now coupled in the x' and y' directions. They can be uncoupled by defining yet another rotating coordinate system X, Y which rotates **clockwise** around z' with an angular frequency of $-\Omega = -\omega\sin\lambda$

$$x' = X\cos\Omega t + Y\sin\Omega t$$

$$y' = -X\sin\Omega t + Y\cos\Omega t$$



Substituting this transformation into the first equation for d^2x'/dt^2:

$$d^2x'/dt^2 = -(g/L)x' + 2\Omega dy'/dt$$

becomes

$$d^2(X\cos\Omega t + Y\sin\Omega t)/dt^2 = -(g/L)(X\cos\Omega t + Y\sin\Omega t) + 2\Omega d(-X\sin\Omega t + Y\cos\Omega t)/dt$$

or

$$d/dt\{ dX/dt\cos\Omega t - X\Omega\sin\Omega t + dY/dt\sin\Omega t + Y\Omega\cos\Omega t\} = -(g/L)(X\cos\Omega t + Y\sin\Omega t) + 2\Omega\{-dX/dt\sin\Omega t - X\Omega\cos\Omega t + dY/dt\cos\Omega t - Y\Omega\sin\Omega t\}$$

Dropping terms of order Ω^2

$$(d^2X/dt^2 + gX/L)\cos\Omega t + (d^2Y/dt^2 + gY/L)\sin\Omega t = 0.$$

Each of the coefficients of $\cos\Omega t$ or $\sin\Omega t$ must vanish for arbitrary time, leaving just the usual simple harmonic motion equations for X, Y:

$$d^2X/dt^2 + (g/L)X = 0$$
$$d^2Y/dt^2 + (g/L)Y = 0$$

That these equations give the usual SHM (including the period $2\pi\sqrt{L/g}$) is really not very surprising. What is important to note is that the XY coordinates rotate with an angular frequency $\omega\sin\lambda = \Omega$ with respect to $x'y'$: that is, the plane of oscillation rotates with respect to a coordinate system on the Earth’s surface. Using

$$\omega = 2\pi / T$$

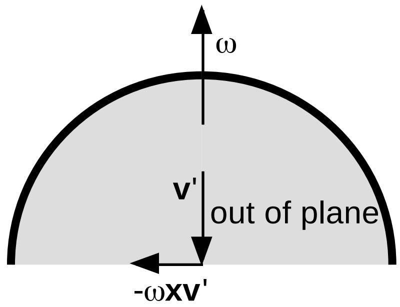
the period of rotation $T_{foucault}$ is then

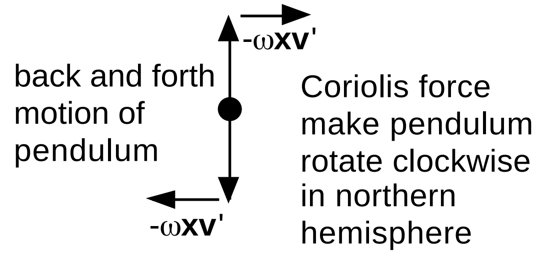
$$T_{foucault} = T_{earth} / \sin\lambda \qquad \text{where } T_{earth} = 24 \text{ hours.}$$

Examples:

- North pole, $\lambda = \pi/2$, $T' = 24$ hours
- Equator, $\lambda = 0$, $T_{foucault} = \infty$ (no rotation)
- $\lambda = 45$ degrees, $T_{foucault} = 24 / (1/\sqrt{2}) = (\sqrt{2})(24) = 34$ hours.

The effect was first demonstrated by French physicist Jean Foucault in 1851; pendulum rotates clockwise in northern hemisphere





Effects on surface winds: