Lecture 20 - Rigid-body motion

Text: similar to Fowles and Cassiday, Chap. 8

We began our discussion of the mechanics of extended objects by introducing the concept of centre-of-mass, and by showing how aspects of dynamics could be separated into motion of the centre-of-mass, and relative motion around the cm position. Here, we take these ideas further by discussing the mechanics of rigid bodies: objects that have a distribution of masses whose shape or relative position is fixed. First, we recap some ideas about angular motion from first year physics.

Angular motion

We have already introduced the angular variable $\theta(t)$ and its derivatives:

angular speed	$\omega = d\theta / dt$
angular acceleration	$\alpha = d\omega / dt$

The angular velocity and acceleration can be related to their linear cousins through relations like:

$$\mathbf{v} = \omega \mathbf{\ x\ r}$$

$$\mathbf{a}_{\text{tangential}} = \alpha \mathbf{\ x\ r}$$

In addition to these kinematical quantities, there are angular analogues of momentum and force:

angular momentum	$L = I\omega$	(not general)
torque	$\tau = I\alpha$	(not general)

where I is the moment of inertia. For a single point mass m, $I = mr^2$.

Again, the linking equations have the form

$$\mathbf{L} = \mathbf{r\ x\ p}$$

$$\tau = \mathbf{r\ x\ F}$$

where **p** is the linear momentum and **F** is the force.

Centre-of-mass

In lecture 19 we introduced the concept of centre-of-mass position $\mathbf{r}_{cm}$ for a distribution of point masses:

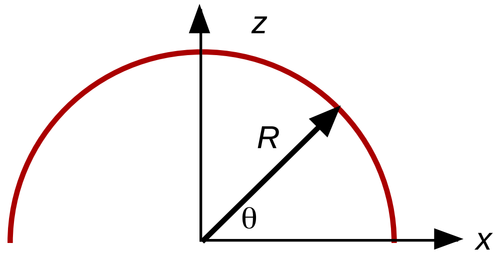
$$\mathbf{r}_{cm} = m_{tot}^{-1} \sum_i m_i \mathbf{r}_i.$$

This can be generalized to a continuous distribution of masses by formally writing

$$\mathbf{r}_{cm} = m_{tot}^{-1} \int \mathbf{r}\, dm. \quad (1)$$

The meaning of the integration variable dm is a little clearer when seen as $\rho(r) d^n r$, where $\rho(r)$ is a position-dependent mass density, describing the mass per unit length, area or volume, corresponding to n = 1, 2 or 3, respectively. Some examples:

Hemispherical wire loop



Recognizing the symmetry of the loop, we place the origin of the coordinate system on the symmetry axis so that $x_{cm} = 0$. For a radius R, the wire has a mass per unit length

$$\rho = m / \pi R.$$

The integral can be performed most easily using the polar angle θ, with

$$dm = \rho R \, d\theta,$$

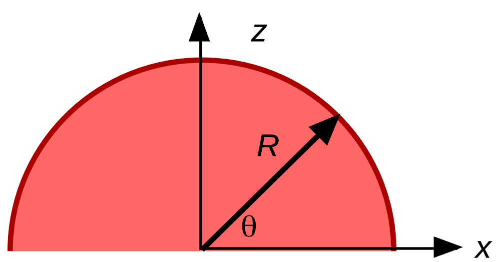
from the arc length $R\,d\theta$. The integral runs from $\theta = 0$ to $\theta = \pi$.

$$\begin{aligned} m\, z_{cm} &= \int_o^\pi z\, dm \\ &= \int_o^\pi z\, \rho R\, d\theta \\ &= \rho R \int_o^\pi R \sin\theta \;\, d\theta \\ &= - \rho R^2 \cos\theta \,\Big|_o^\pi \\ &= \;2\rho R^2. \end{aligned}$$

Substituting $\rho = m / \pi R$, the cm position becomes

$$z_{cm} = 2R / \pi.$$

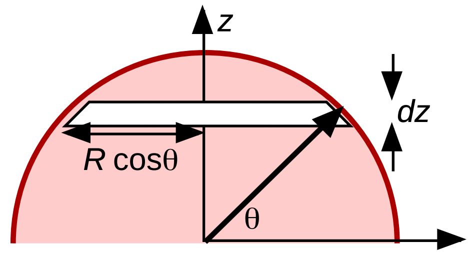
Semicircular lamina



Now we take the same shape as the first example, but fill in the frame to form a two-dimensional sheet. The density is now a mass per unit area, given by

$$\rho = m / (\pi R^2/2).$$

The integral is most easily performed again using θ as the integration variable. The mass element dm is a slice of the lamina at constant z, so θ runs from 0 to $\pi/2$:



The area of the slice is $2R\cos\theta\, dz$. We convert from dz to $d\theta$ by using

$$dz / d\theta = d\,(R\sin\theta) / d\theta = R\cos\theta$$

so that the area becomes

$$2R^2 \cos^2\theta\, d\theta$$

and the mass element is

$$dm = 2\rho R^2 \cos^2\theta\, d\theta$$

Finally, the position of the cm can be found from

$$m\, z_{cm} = \int z\, dm = \int_0^{\pi/2} R\sin\theta \bullet 2\rho R^2 \cos^2\theta\, d\theta$$
$$= 2\rho R^3 \int_0^{\pi/2} \sin\theta \;\cos^2\theta\, d\theta$$

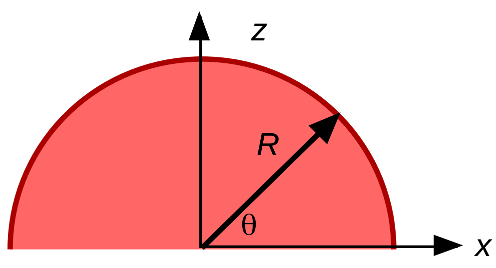
But $\int_0^{\pi/2} \sin\theta \;\cos^2\theta\, d\theta = \int_0^{1} \cos^2\theta\, d\cos\theta = x^3/3\,|_0^1 = 1/3$
so

$$m\, z_{cm} = 2\rho R^3/3.$$

Lastly, substituting for the density $\rho = m / (\pi R^2/2)$ gives

$$z_{cm} = (2R^3/3) / (\pi R^2/2)$$
$$= 4R / 3\pi.$$

Solid hemisphere



As a last example, we take a solid, three-dimensional sphere and cut it in half. The density is now a mass per unit volume, given by

$$\rho = m / (2\pi R^3/3).$$

Since we're so practiced, we again use θ as the integration variable. The mass element dm is a slice of the sphere at constant z, so θ runs from 0 to $\pi/2$ as with the lamina. But the volume of the slice (as opposed to the area of the lamina) is

$\pi (R\cos\theta)^2 \, dz$.

Converting from dz to $d\theta$ by using (as before) $dz / d\theta = d(R\sin\theta) / d\theta = R\cos\theta$ the volume becomes

$\pi (R\cos\theta)^3 \, d\theta$

and the mass element is

$dm = \rho\pi (R\cos\theta)^3 \, d\theta$

Finally, the position of the cm can be found from

$$m\, z_{cm} = \int z \, dm = \int_0^{\pi/2} R\sin\theta \bullet \rho\pi (R\cos\theta)^3 \, d\theta$$
$$= \pi\rho R^4 \int_0^{\pi/2} \sin\theta \, \cos^3\theta \, d\theta$$

But $\int_0^{\pi/2} \sin\theta \, \cos^3\theta \, d\theta = \int_0^1 \cos^3\theta \, d\cos\theta = x^4/4 \,|_0^1 = 1/4$, so

$$m\, z_{cm} = \pi\rho R^4 /4.$$

Lastly, substituting for the density $\rho = m / (2\pi R^3/3)$ gives

$$z_{cm} = (\pi R^4 /4) / (2\pi R^3/3)$$
$$= 3R / 8.$$

Summary

Note how the cm decreases as the mass is shifted to the base of the shape in 1D to 3D

Shape	z_{cm}/R	
ring	$2/\pi$	0.64
lamina	$4/3\pi$	0.42
hemisphere	3/8	0.375