Lecture 12 - Energy and temperature

What's important:
- Einstein's energy equation
- binding energy
- Q-value
- introduction to temperature

Text: Carroll and Ostlie, Sec. 4.4
PHYS 120 on-line: *Modern Physics: from Quarks to Galaxies*

Energy-momentum equation

According to Einstein, a particle's energy E is related to its mass m and momentum p through

$$E^2 = p^2c^2 + m^2c^4, \qquad (12.1)$$

where c is the speed of light, and has a value of 3.00 x 10^8 m/s. Note:
- this equation is quadratic in E, p and m
- it depends on momentum p, not velocity
- it has a contribution from a mass energy mc^2.

Where does kinetic energy fit into this? By definition:

$$E = mc^2 + K.$$

The relation $K = 1/2\ mv^2$ only applies at small speeds, as can be verified by expanding the energy expression at small v/c:

$$E = (p^2c^2 + m^2c^4)^{1/2} = mc^2\,(1 + p^2c^2/\,m^2c^4)^{1/2}$$
$$\cong mc^2\,[1 + (1/2)p^2/\,m^2c^2] = mc^2 + p^2/\,2m. \qquad \text{(at small } p/mc\text{)}$$

Binding energy

Under some circumstances, interacting particles can form stable bound states: for example, protons and neutrons can form a nucleus ^{2}H (often referred to as the deuteron D):

$$\mathrm{p} + \mathrm{n} \rightarrow {}^2\mathrm{H}.$$

We know that energy is released when a bound state is formed, and we know that it takes energy to break up a bound state. From the general energy-momentum equation, the release in energy when a bound system is formed must come from a decrease in the mass energy mc^2 of the system. In other words, *a bound system has less mass energy than its unbound components have in isolation*.

The binding energy *B.E.* is alternatively:
- the amount of energy released when the system forms from its unbound components

- the amount of energy that must be added to the system to break it into unbound components at rest.

Hence, *B.E.* is the difference in mass energies between the bound state and the isolated components of the bound state. In symbols:

$$B.E. = \Sigma_i \, m_i c^2 - m_{\text{total}} c^2$$

where the sum over the individual components *i* refers to the masses of the particles in isolation.

For atoms and nuclei, it is more convenient to use electron-volts (eV) as an energy unit, rather than Joules.

$$1 \text{ eV} = 1.6 \times 10^{-19} \text{ J} \qquad 1 \text{ MeV} = 10^6 \text{ eV} = 1.6 \times 10^{-13} \text{ J}$$

Example The deuteron (or ^{2}H) consists of a proton and a neutron and has a binding energy of 2.23 MeV (or 3.6×10^{-13} J). What is the decrease in mass when this nucleus forms from a proton and neutron?

From $B.E. = \Delta m c^2$,

$$\Delta m = B.E. / c^2 = 3.6 \times 10^{-13} / (3.0 \times 10^8)^2 = 4 \times 10^{-30} \text{ kg.}$$

The fractional change in mass is then

$$\frac{\Delta m}{m_p + m_n} = \frac{4 \times 10^{-30}}{2 \bullet 1.67 \times 10^{-27}} = 0.1\%$$

The example shows that the fractional change in mass in the formation of a nucleus is small, but measurable. But the deuteron is a loosely bound system, and for most nuclei, the fractional change is closer to 1%.

Q-value

When particles interact, they can exchange energy and their mass may change. If they simple exchange energy, then the kinetic energy of the pair is conserved. However, if their total mass changes, then the total kinetic energy changes as well. The *Q*-value is defined as the change in the mass energy of the system in a reaction:

$$Q\text{-value} = \Sigma_{\text{initial}} \, m_i c^2 - \Sigma_{\text{final}} \, m_f c^2$$

where *i* and *f* refer to all the particles in the initial and final states (we write out *Q*-value, rather than just *Q*, to distinguish it from the charge *Q*). With this choice of sign, the *Q*-value is positive if the initial mass energy is larger than the final mass energy, thus releasing energy to the system.

This definition can be applied to decays as well:

$$Q\text{-value} = m_{\text{initial}} c^2 - \Sigma_{\text{final}} \, m_f c^2$$

For a decay to occur, its *Q*-value cannot be negative.

Example How much energy is released if two protons and two neutrons fuse to form helium?

For the reaction 2p + 2n → ^{4}He + γ, the *Q*-value is

$$Q = 2m_pc^2 + 2m_nc^2 - m_{He}c^2$$
$$= 2m_pc^2 + 2m_nc^2 - [\ 2m_pc^2 + 2m_nc^2 - B.E.(^4\mathrm{He})\ ]$$
$$= B.E.(^4\mathrm{He}).$$

Substituting for the binding energy of ^{4}He, we find

$$Q = 28.2959 \text{ MeV} = 4.55 \times 10^{-12} \text{ J}.$$

Temperature and energy

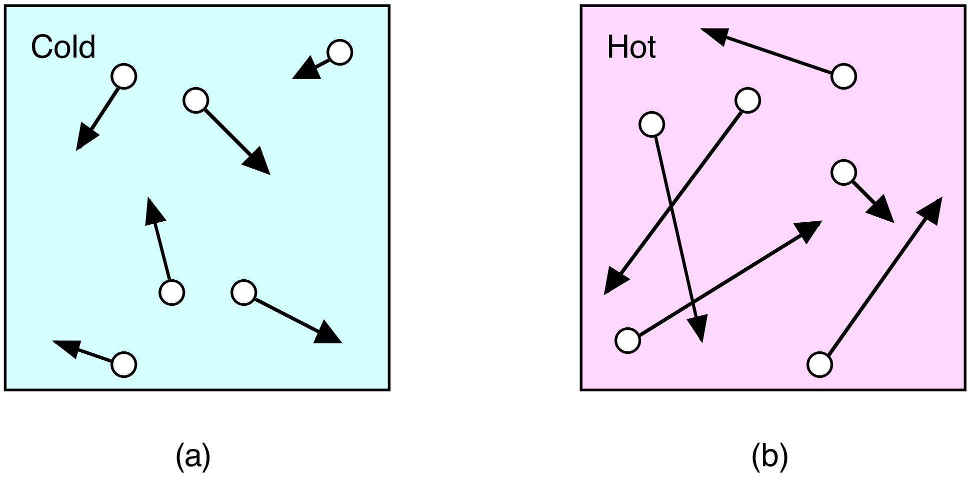
It has been known for almost two centuries that gases at low density obey the ideal gas equation,

$$PV \propto T, \qquad (12.2)$$

where the gas in question is subject to a pressure *P*, occupies a volume *V* and has an absolute temperature *T*. Pressure has units of energy density, and the left hand side of Eq. (12.2) is proportional to (but not equal to!) the kinetic energy of the gas, so that the kinetic energy of an atom or molecule in the gas is proportional to the temperature:

$$[\textit{kinetic energy}] \propto T.$$

Consider a gas in a fixed container:



(a) (b)

If we could determine the kinetic energy of every molecule at every instant in time we would find:

- not all molecules have the same kinetic energy at the same time. Some molecules are moving slowly, some fast, so the molecules have a *distribution* of kinetic energies.
- a given molecule exchanges energy with its surroundings through collisions, so the kinetic energy of each molecule changes with time.

The functional form of the kinetic energy distribution was found by Maxwell in 1859, and the time evolution of such distributions was described by Boltzmann in 1872. For an ideal gas in a three-dimensional box the average kinetic energy per particle is

[*average kinetic energy per particle*] = (3/2) $k_B T$ (12.3)

k_B = Boltzmann's constant = 1.38×10^{-23} J/K

$k_B = R / N_o$ (R is gas constant; N_o is Avodagro's number)