Lecture 18 - Reactions I - Tunneling

What's Important:

- Coulomb barrier
- barrier penetration factor
- kinematics

Text: Clayton, Secs. 4.1 and 4.5

Coulomb barrier to nuclear reactions

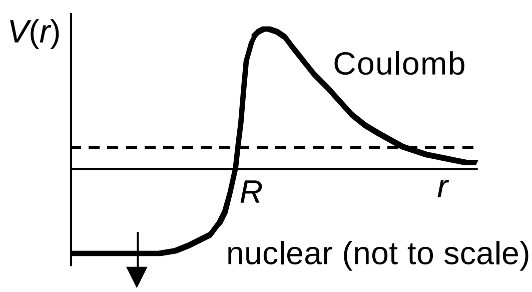
The elementary particle reactions that occurred in the Big Bang took place at elevated temperatures of 10^9 K, corresponding to kinetic energies of MeV. In contrast, the energy scale of stars is a hundred times lower, set by a temperature scale of a few times 10^7 K. Such low temperatures affect reactions in two ways:

- the reactions may occur through the phenomenon of quantum mechanical tunneling
- the low energy end of a thermal distribution of particles may not participate in a reaction.

In this lecture, we provide the conceptual framework for tunneling phenomena, and incorporate thermal effects in the following two lectures.

The potential energy experienced by a positively charged particle approaching a nucleus has the following schematic form:

- short distance: strong interaction $V \sim -50 \pm 10$ MeV
- long distance: Coulomb repulsion $V(r) = Q_1Q_2 / (4\pi\varepsilon_o r) = 1.44\, Z_1Z_2/r$ (MeV) where r is quoted in fm.



The change in the potential occurs at about $R = 1.4\, A^{1/3}$ fm, which is slightly larger than the commonly quoted value for the nuclear mass distribution at $1.2\, A^{1/3}$ fm.

Example: Compare the barrier at the nuclear boundary with the typical thermal energy of a particle at $T = 10^7$ K for the reaction p + ^{12}C.
Barrier: $Z_1 = 1$, $Z_2 = 6$ $R = 1.4 \times 12^{1/3} = 3.2$ fm $V(R) = 1.44 \bullet 1 \bullet 6/3.2 = 2.7$ MeV
Energy: $k_BT = 1.38\text{x}10^{-23} \bullet 10^7 / 1.6\text{x}10^{-19} = 860$ eV = 0.86 keV.

Clearly, the typical energy of a particle in the stellar interior is much less than the barrier height for reactions involving light nuclei. The dashed line on the diagram doesn't do justice to just how different the energy scales are.

Quantum mechanical tunneling

Classically, a particle would be forbidden from passing through the barrier in a situation illustrated by the diagram, where the relative kinetic energy is a thousand times lower than the barrier height. Because of the uncertainty relation, the passage is allowed by quantum mechanics, with a probability P defined by

$$P \equiv |A_\infty|^2 / |A_R|^2 \qquad (18.1)$$

where the (complex) particle amplitudes are A_∞ at infinite separation and A_R at the nuclear edge $r = R$ (18.1 applies to a particle initially trapped at $r < R$).

Calculating the penetration factor for square barriers is a problem performed in second year courses introducing quantum concepts. In third year, one shows how to obtain P for continuous potentials by integrating over the barrier shape between the classical turning points, the so-called WKB approximation. Here, the nuclear + Coulomb potential has three pieces when written in terms of the separation r:

- the nuclear piece, essentially a square well of constant depth V = -50 MeV
- the Coulomb piece, which decreases like $1/r$
- an orbital piece arising in three dimensions by separating out the angular variables:

$$V_{\text{ORBITAL}} = \frac{\ell(\ell+1)\hbar^2}{2\mu r^2} \qquad \mu = \text{reduced mass}, \ \ell = \text{orbital ang. momentum}$$

The orbital contribution to the effective potential is repulsive, so the greatest likelihood of tunneling is for $\ell = 0$ states. The full solution for arbitrary ℓ is given in Clayton's book on stellar evolution; here we only quote the result for $\ell = 0$, the dominant pathway:

$$P_o \sim (E_C/E)^{1/2} \exp(-W_o)$$

where the subscript specifies the angular momentum state. Here,

E is the relative kinetic energy

E_C is the Coulomb barrier $E_C = \left(\frac{e^2}{4\pi\varepsilon_o}\right)\frac{Z_1 Z_2}{1.4(A_1^{1/3} + A_2^{1/3})}$ (touching spheres)

and

$$W_o = \frac{4Z_1Z_2}{\hbar v}\left(\frac{e^2}{4\pi\varepsilon_o}\right)\left[\frac{\pi}{2} - \sin^{-1}\left(\frac{E}{E_C}\right)^{1/2} - \left(\frac{E}{E_C}\right)^{1/2}\left(1 - \frac{E}{E_C}\right)^{1/2}\right].$$

In this last expression, v is the relative velocity of the pair of reacting particles.

For the small values of E/E_C applicable to stellar interiors, the only term of importance in the square bracket is $\pi/2$, leading to the approximate expression for W_o:

$$W_o = \frac{2\pi Z_1 Z_2}{\hbar v}\left(\frac{e^2}{4\pi\varepsilon_o}\right).$$

Now, the relative velocity v is proportional to the square root of the kinetic energy through

$$E = \mu v^2/2 \quad \text{-->} \quad v = (2E/\mu)^{1/2},$$

so we can write

$$W_o = b\,E^{-1/2} \quad \text{with} \quad b = \sqrt{\frac{\mu}{2}}\,\frac{2\pi Z_1 Z_2}{\hbar}\left(\frac{e^2}{4\pi\varepsilon_o}\right).$$

Just to recap, we approximate the leading order contribution to the barrier penetration probability by

$$P_o = \left(\frac{E_C}{E}\right)^{1/2}\exp(-bE^{-1/2}).$$

Q-values

We finish off this lecture with some short comments about Q-values. Our notation, useful for nuclear reactions, is to write a reaction in the form

$$a + X \rightarrow Y + b$$

where a,b are light particles or nuclei and X,Y are heavier nuclei. Conservation of energy requires

$$E_{aX} + (m_a + m_X)c^2 = E_{bY} + (m_b + m_Y)c^2,$$

where

E_{ij} = the kinetic energy of the ij system.

Clearly, we have to be concerned about the mc^2 energy of the reactants. Now, atomic masses (nuclei + electrons!) are often tabulated in terms of the mass excess Δm

$$m\,(Z,N) \equiv \Delta m\,(Z,N) + Am_u,$$

where m_u is the atomic mass unit

m_u = 931.478 MeV.

Because baryon number (essentially A) is conserved, the energy balance equation can be written as

$$E_{aX} + (\Delta m_a + \Delta m_X)c^2 = E_{bY} + (\Delta m_b + \Delta m_Y)c^2.$$

This equation can be juggled a bit to write

$$Q\text{-value} = (\Delta m_a + \Delta m_X - \Delta m_b - \Delta m_Y)c^2.$$