Lecture 20 - Reactions III - Thermonuclear processes

What's Important:
- energy-dependent cross sections
- complete rate equation

Text: Carroll and Ostlie, Sec. 10.3
Clayton, Secs. 4.3 and 4.6

Energy-dependent cross section

In the previous two lectures, we showed that the reaction rate is proportional to the thermal expectation $<\sigma v>$, where the cross section σ may be energy-dependent. We demonstrated how to take the thermal average <...> using the Maxwell-Boltzmann distribution of velocities. We now return to the cross section, and then complete our derivation of the rate equation.

We have already established that the probability of tunneling through a Coulomb barrier at zero angular momentum is

$$P \sim E^{-1/2} \exp(-b/\sqrt{E}). \qquad (20.1)$$

How does P affect the cross section σ? If P were constant, we would expect σ to be proportional to the square of the de Broglie wavelength (giving units of area) just like the scattering of water waves on an isolated rock

$$\sigma \sim [wavelength]^2 \sim (h / mv)^2 \sim 1/E. \qquad (20.2)$$

The function $\exp(-b/\sqrt{E})$ rises from 0 to 1 with increasing E, suggesting that we parametrize the reaction cross section as

$$\sigma(E) = \frac{S(E)}{E} \exp(-b/\sqrt{E}), \qquad (20.3)$$

where $S(E)$ is taken to be a slowly-varying function that varies from one reacting pair to the next, and must be obtained experimentally.

(*Aside: those with some familiarity of resonant states know that there will be kinematic regions where 20.3 is not valid*).

This equation gives us the energy-dependence of σ, but the expectation is expressed in terms of velocities. Let's make the transformation to a common integration variable, then evaluate $<\sigma v>$.

Using

$$v = \left(\frac{2E}{\mu}\right)^{1/2} \quad \text{and} \quad dv = \frac{1}{2}\left(\frac{2}{\mu E}\right)^{1/2} dE, \qquad (20.4)$$

then the distribution

$$\phi(v) = 4\pi v^2 \left(\frac{\mu}{2\pi k_B T}\right)^{3/2} \exp\left(-\frac{\mu v^2}{2k_B T}\right)$$

leads to

$$\phi(v)dv = 4\pi v^2 \left(\frac{\mu}{2\pi k_B T}\right)^{3/2} \exp\left(-\frac{\mu v^2}{2k_B T}\right) dv$$

$$= 4\pi \left(\frac{\mu}{2\pi k_B T}\right)^{3/2} \exp\left(-\frac{E}{k_B T}\right) v^2 dv$$

$$= 4\pi \left(\frac{\mu}{2\pi k_B T}\right)^{3/2} \exp\left(-\frac{E}{k_B T}\right) \frac{2E}{\mu} \frac{1}{2} \left(\frac{2}{\mu E}\right)^{1/2} dE$$

$$= \frac{2}{\sqrt{\pi}} \frac{E}{k_B T} \exp\left(-\frac{E}{k_B T}\right) \frac{dE}{(k_B T E)^{1/2}} \equiv \Psi(E)\, dE$$

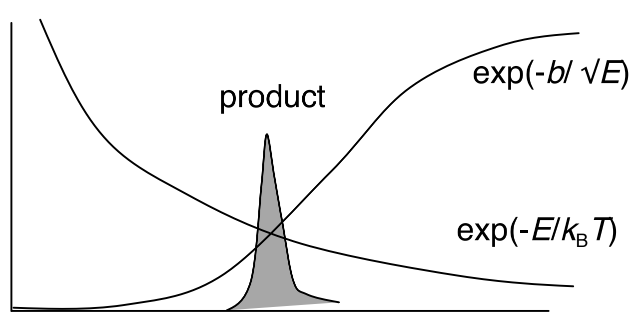
The expectation $<\sigma v>$ is then

$$< \sigma v > = \int \sigma(E) \quad v \quad \Psi(E) \quad dE$$

$$= \int \frac{S(E)}{E} \exp\left(-\frac{b}{\sqrt{E}}\right) \quad \sqrt{\frac{2E}{\mu}} \quad \frac{2}{\sqrt{\pi}} \frac{E}{k_B T} \exp\left(-\frac{E}{k_B T}\right) \frac{dE}{(k_B T E)^{1/2}} \qquad (20.5)$$

$$= \left(\frac{8}{\pi \mu}\right)^{1/2} \frac{1}{(k_B T)^{3/2}} \int S(E) \exp\left(-\frac{E}{k_B T} - \frac{b}{\sqrt{E}}\right) dE$$

Of the two pieces in the integrand, $S(E)$ varies slowly and the exponential varies rapidly. In fact, the exponential involves the product of two terms, one of which is very small at large E and the other of which is very small at small E:



The product occurs when the argument of the exponential is at a MINIMUM (because the function is of the form exp(-argument)]. We can find the minimum of the argument

$E / k_B T + b / \sqrt{E}$

by equating to zero its derivative with respect to E:

$$0 = \frac{d}{dE}\left(\frac{E}{k_B T} + \frac{b}{\sqrt{E}}\right) = \frac{1}{k_B T} - \frac{1}{2}\frac{b}{E^{3/2}}$$

which is satisfied by the value E_o of

$$E_o = (bk_B T / 2)^{2/3}.$$

Inserting the value of b calculated in a previous lecture, and invoking the notation $T = T_6 \cdot 10^6$ (K),

$$E_o = 1.22\ (Z_1\ Z_2\ A^{1/2}\ T_6)^{2/3} \quad \text{(keV)}$$

where A is the (dimensionless) reduced mass:

$$A = A_1 A_2 / (A_1 + A_2).$$

This E_o is called the *most effective energy* for thermonuclear reactions, and is in the range 10-30 keV for light nuclei, a good factor of ten larger than $k_B T$ as calculated previously.

The integral can thus be approximated by

$$\Lambda = 7.20\text{x}10^{-25}\ S_o\ K^2\ e^{-K} / (AZ_1 Z_2) \qquad \text{m}^3/\text{s}$$

where $S_o \equiv S(E_o)$ and is quoted in keV-barns (1 barn = 10^{-24} cm^2 = 10^{-28} m^2), and the dimensionless constant K is

$$K = (3E_o / k_B T) = 42.48\ (Z_1^2 Z_2^2 A / T_6)^{1/3}.$$

Lastly, defining the mass fraction X_i of species i by

$$X_i = \frac{\textit{mass of i per unit volume}}{\textit{total mass per unit volume}} = \frac{A_i N_i M_u}{\rho}$$

where N_i is the number density of species i, M_u is the atomic mass unit (931... MeV) and ρ is the mass density (consistent with the units of M_u).

Hence, the rate equation is

$$r_{12} = \frac{2.62 \times 10^{29}}{(1+\delta_{12}) A Z_1 Z_2}\rho^2 \frac{X_1 X_2}{A_1 A_2} S_o K^2 e^{-K} \qquad \text{m}^{-3}\text{s}^{-1}.$$

(the prefactor is the same if ρ and r_{12} are in MKSA or cgs units). This is an approximation to the correct result, but is more than adequate for our purpose here. See p. 300-360 of Clayton for a further discussion and correction terms.