Lecture 24 - Stellar interiors

What's Important:
- interior temperature and pressure
- stability conditions

Text: Carroll and Ostlie, Secs. 10.1, 10.2

Stellar interiors

As a gas cloud (hydrogen and helium, for example) collapses under gravity to form a star, some of its gravitational potential energy is released to molecular kinetic energy, raising the temperature of the gas. The resulting pressure from the hot, ionized hydrogen cloud, and its accompanying photon gas, resists the further condensation of the cloud. Because the density-temperature profile of a star is not constant as a function of radial distance, its determination must be performed by numerical techniques. Nevertheless, we can at least outline the principles involved, and provide crude estimates of the conditions in a star's interior.

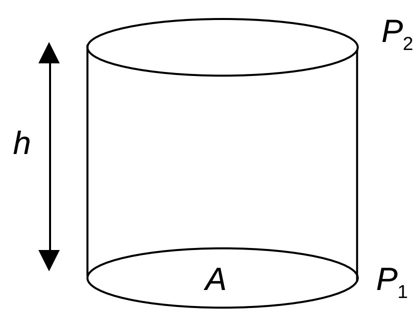
Hydrostatic equilibrium

The pressure in a star is not constant, but decreases away from its centre because of gravity, much like the pressure of the Earth's atmosphere is greatest at the Earth's surface, and decreases with height. Thus, there is a pressure gradient as a function of height. As shown in some first year courses (at SFU, PHYS 101 but not PHYS 120), the pressure difference between two locations separated by a vertical distance h is

$$P_1 - P_2 = \rho g h$$

where ρ is the local *mass* density and g is the local gravitational acceleration. Let's first review the proof to re-establish our familiarity with fluids.

Consider the force experienced by an area element A at the lower surface in the following diagram (the cylinder is just a mathematical surface chosen for convenience):



.
The total pressure P_1 is equal to the sum of the pressure at the upper surface P_2 and the weight per unit area mg/A of the column of fluid.

We obtain the weight from the effective gravitational acceleration g times the mass of the fluid in the mathematical column. Now the mass of this column is

[*mass*] = [*density*] • [*volume*]

or

$$m = \rho Ah$$

Thus

$$P_1 = P_2 + mg / A = P_2 + \rho Ahg / A$$

or

$$P_1 - P_2 = \rho gh. \qquad (24.1)$$

Taking the infinitesimal limit $h \rightarrow 0$, this becomes

$$dP / dh = -\rho g, \qquad (24.2)$$

where the minus sign arises because P decreases as h increases.

Now, let's assume that the mass distribution is spherically symmetric. At a radial position r, g depends only on the mass enclosed by a mathematical surface at r, not on the mass outside of r. Again, from first year physics,

$$g = G M_r / r^2.$$

Further, if the enclosed density is uniform,

$$M_r = (4\pi/3) \rho r^3,$$

and we predict

$$g \sim r^1 \qquad \text{(uniform sphere).}$$

That is, the effective acceleration is lowest at a star's center, then increases with the distance from the center.

Estimated pressure in the Sun

As a really crude estimate of the pressure P_{core} at the Sun's core, let's assume that dP/dh is constant, with zero pressure at $h = R_{Sun} = 6.96 \times 10^8$ m. Thus

$$dP/dh = -P_{core}/R_{Sun}$$

so that

$$P_{core}/R_{Sun} = \rho g.$$

We'll take g to be constant, given by

$$g = GM_{Sun}/R_{Sun}^2$$

which gives

$$P_{core} = \rho GM_{Sun}/R_{Sun}.$$

Now, the mass density of the Sun is 1.41×10^3 kg/m^3 if its mass is spread equally over it volume. Thus

$P_{core} = 1.41 \times 10^3 \bullet 6.67 \times 10^{-11} \bullet 1.99 \times 10^{30} / 6.96 \times 10^8$
$= 2.7 \times 10^{14}$ J/m^3.

While crude, this calculation provides the correct order of magnitude for the pressure in the core. To put the numerical value into perspective, the pressure is 2.7×10^9 atm!

Contributions to the pressure

The nuclei and electrons of an ionized gas approximately obey the ideal gas equation

$$PV = Nk_BT$$

or

$$P = nk_BT, \quad (24.3)$$

where n is the NUMBER density, not the number of moles as it is in chemical applications.

As established in the previous lecture, the photon pressure P_γ is given by

$$P_\gamma = U_\gamma /3, \quad (24.4)$$

where the photon energy density U_γ is given by

$$U_\gamma = 7.565 \times 10^{-16}\, T^4 \quad \text{J/m}^3 \quad (T \text{ in K}). \quad (24.5)$$

Two simple calculations will suffice to give the relative order of magnitudes for these contributions in the Sun's interior at $T = 15 \times 10^6$ K.

Ideal gas
Spreading the Sun's 10^{57} baryons uniformly over its volume gives a number density of 1×10^{30} m^{-3}, which becomes

$$n \sim 2 \times 10^{30} \text{ m}^{-3}$$

when electrons are taken into account. Thus, the ideal gas pressure under this assumption is

$$P = 2 \times 10^{30} \bullet 1.38 \times 10^{-23} \bullet 1.5 \times 10^7 = 4 \times 10^{14} \text{ J/m}^3.$$

Photon gas
Only the temperature T is needed for this calculation,

$$P_\gamma = 7.565 \times 10^{-16}\, T^4 / 3 = 7.565 \times 10^{-16}\, (1.5 \times 10^7)^4 / 3 = 1.3 \times 10^{13} \text{ J/m}^3.$$

Thus, we see that the pressure from the photon gas is less than 10% that from the nuclei and electrons.

Summary

We see from the previous calculation that an assumed core temperature of 10^7 K gives the correct pressure to support the Sun. The many approximations of this calculation can be dropped by doing the following:

- Use the mass and radius of the Sun to determine P_{core} under model conditions by integrating Eq. (24.2).
- Use Eqs. (24.3) and (24.4) to determine the temperature distribution.
- Predict the energy output using the PP chains for the relevant temperatures.
- Iterate as needed.

The net result of such model calculations for the Sun is satisfactory (standard solar model) and permits the prediction of the neutrino flux from the Sun's nuclear reactions. The measured flux, extensively measured, is low by a factor of three and is called the solar neutrino problem. Recent measurements of neutrinos from the Sun have suggested new properties for the neutrino, possibly associated with a very small non-vanishing mass.