Lecture 26 - A star's birth

What's Important:

- time scales for collapse

Text: Carroll and Ostlie, Sec. 12.2

Time scales for gravitational collapse

The time scale for the collapse of a gas cloud to form a star can be determined from Newton's laws. The solutions are usually numerical, although some analytical results can be obtained under a few simple approximations. The contraction process has similarities to the expansion of the universe treated earlier in this course, where we established that the characteristic expansion time is of the order

$$t_{exp} \sim (8\pi G\rho/3)^{-1/2}. \qquad (26.1)$$

Let's assume that particles in the contracting cloud are under free fall conditions, and face no pressure gradients from their surroundings, only gravitational attraction from the cloud's center. Then, for a particle of mass m lying a distance r from the center of the cloud, Newton's second law reads

$$\frac{d^2r}{dt^2} = -G\frac{M_r}{r^2}, \qquad (26.2)$$

after canceling a common factor of m from both sides of the equation (the effects of pressure gradients are discussed in Chap. 10 of Carroll and Ostlie). The enclosed mass M_r does not change during the contraction (an outer shell does not overtake an inner shell), so its value is set at the beginning of the contractive phase by the radius r_o:

$$M_r = (4\pi/3)\rho_o r_o^3, \qquad (26.3)$$

where ρ_o is the initial density of the cloud. We can find the velocity at position r either by integrating Eq. (26.2), or by using potential energy (where the integral has already been done for us). Equating the gain in kinetic energy with the loss in gravitational potential energy

$$\Delta K = -\left(Gm\frac{4\pi}{3}\rho_o r_o^3\right)\left(\frac{1}{r_o} - \frac{1}{r}\right)$$

$$= \left(Gm\frac{4\pi}{3}\rho_o r_o^3\right)\left(\frac{1}{r} - \frac{1}{r_o}\right).$$

Choosing the initial kinetic energy at $r = r_o$ to be zero, then

$$\Delta K = mv^2/2$$

and

$$\frac{1}{2}v^2 = \left(\frac{4\pi}{3}G\rho_0 r_0^3\right)\left(\frac{1}{r} - \frac{1}{r_o}\right)$$

so that

$$v = -\left[\frac{8\pi}{3} G\rho_0 r_0^2 \left(\frac{r_0}{r} - 1\right)\right]^{1/2}, \tag{26.4}$$

where the minus sign is chosen to recognize that the material moves towards smaller values of r.

Eq. (26.4) gives the velocity as a function of position, not as a function of time. Integrating this relationship to obtain position as a function of time requires a change of variables to cast the equation into something other than an inverse square root relation. First, we introduce two new variables

$$\chi \equiv r / r_o$$

and

$$\kappa \equiv (8\pi G\rho_o/3)^{1/2}.$$

Notice the immediate similarity between Eq. (26.1) and the definition of κ. With these definitions, Eq. (26.4) becomes

$$r_0 \frac{d(r / r_0)}{dt} = -r_0 \left(\frac{8\pi G\rho_0}{3}\right)^{1/2} \left(\frac{r_0}{r} - 1\right)^{1/2}$$

or

$$\frac{d\chi}{dt} = -\kappa \left(\frac{1}{\chi} - 1\right)^{1/2}. \tag{26.5}$$

This looks a little cleaner. The next step involves trigonometric substitutions, starting with the replacement

$$\chi = \cos^2\xi, \tag{26.6}$$

such that Eq. (26.5) becomes

$$LHS = \frac{d\chi}{dt} = \frac{d\cos^2\xi}{dt} = -2\cos\xi \sin\xi \frac{d\xi}{dt}$$

$$RHS = -\kappa \left(\frac{1}{\chi} - 1\right)^{1/2} = -\kappa \left(\frac{1}{\cos^2\xi} - 1\right)^{1/2} = -\kappa \frac{\sin\xi}{\cos\xi}$$

or

$$\cos^2\xi \frac{d\xi}{dt} = \frac{\kappa}{2}. \tag{26.7}$$

The right hand side of this expression is a constant, and integrates immediately to

$$\kappa t / 2 + C_2,$$

where C_2 is an integration constant. The left hand side contains the time-dependent variable ξ, and can be integrated using a trigonometric identity

$$\cos 2\xi = \cos^2\xi - \sin^2\xi = \cos^2\xi - (1 - \cos^2\xi) = 2\cos^2\xi - 1$$

whence

$$\cos^2\xi = 1/2 + \cos 2\xi / 2.$$

So

$$\int_0^{\xi} \cos^2\xi \frac{d\xi}{dt} dt = \frac{\xi}{2} + \frac{1}{2} \cdot \frac{1}{2} \sin 2\xi$$

where the lower limit arises from $\chi = 1$ at $r = r_o$, or $\cos^2\xi = 1$ or $\xi = 0$. Thus, the solution is

$$\frac{\xi}{2} + \frac{1}{4}\sin 2\xi = \frac{\kappa}{2} t$$

where an integration constant has been set equal to zero so $\xi = 0$ when $t = 0$. Finally,

$$\xi + \frac{1}{2}\sin 2\xi = \kappa t \qquad (26.8)$$

It's taken us a lot of work to get this far, and what we've ended up with is an equation which must be solved numerically to find r as a function of t, where $t = 0$ when the collapse starts at $r = r_o$. However, our main interest is the time taken for collapse, which we call t_{ff} for the free-fall time (since we have neglected pressure terms *etc.*). This can be obtained from Eq. (26.8) through the extreme situation of $r = 0$. This is not a bad approximation; using the example from the previous lecture, a giant molecular cloud collapses from a number density of 10^{14} m^{-3} to the Sun at 10^{30} m^{-3}, corresponding to a ratio of 10^5 in length scale, or $r / r_o = 10^{-5}$.

Now, $r = 0 \rightarrow \chi = 0$ or $\cos\xi = 0$, corresponding to $\xi = \pi/2$. The left-hand side of Eq. (26.8) is then

$$LHS = \pi/2 + 0.$$

Hence,

$$t_{ff} = \pi / 2\kappa$$

or

$$t_{ff} = \left(\frac{3\pi}{32 G \rho_0}\right)^{1/2} \qquad (26.9)$$

Once again, note the similarity between (26.1) and (26.9). Further, the time scale for the collapse is the same for all shells of the star, independent of r_o.

Example

Estimate the collapse time for the giant molecular cloud example considered in the previous lecture:

$$n = 10^{14} \text{ m}^{-3} \qquad \rightarrow \qquad \rho_o = m_H n = 1.67 \times 10^{-27} \bullet 10^{14} = 1.7 \times 10^{-13} \text{ kg/m}^3$$

$$t_{ff} = \left(\frac{3\pi}{32 \cdot 6.67\times10^{-11} \cdot 1.7\times10^{-13}}\right)^{1/2} = 1.6\times10^{11}\text{ s}$$

or

$$t_{ff} = 5100 \text{ years.}$$

Clearly, this result depends on the choice of ρ_o, and will be longer for clouds of lower density. Typically, the time scale for collapse is of the order 10^5 years, certainly much less than the 10^{10} year lifetime of stars like our Sun.