

Lecture 29 - White dwarfs and neutron stars

What's Important:

- Σ white dwarfs and the Chandrasekhar limit

- Σ neutron stars and pulsars

Text: Carroll and Ostlie, Chap. 15 (ex. 15.5)

White dwarfs

Sirius was observed (by a new 18-inch refractor) to be a binary system in 1862, almost 20 years after it was proposed to be such a system by Friedrich Bessel. The smaller of the binary pair has a mass now quoted at 1.05 ± 0.03 solar masses, and a surface temperature (determined half a century later) of a very hot 27,000 K in spite of a luminosity just 0.03 that of our Sun. These observations can be taken together to paint a picture of an altogether different type of star than the Sun:

- Σ luminosity + temperature give a radius of 0.008 solar radii, about the size of the Earth; thus, the average density is $(0.008)^{-3} = 2,000,000$ times that of the Sun

- Σ temperature + density show that star is not hydrogen, or PP chains would produce a luminosity several orders of magnitude larger than observed.

Sirius B is known as a *white dwarf*, similar mass to the Sun, but much smaller (*dwarf*) and hotter (*white*). Most of the star is ionized carbon and oxygen. The source of the pressure to support a white dwarf comes from its electrons.

In a fermion gas, particles have non-zero momentum even at vanishing temperature, and this motion generates a pressure. Calculating the pressure can be done with standard statistical mechanics techniques, although here we simply quote the result **for a non-relativistic gas at $T = 0$** :

$$P = \frac{(3p^2)^{2/3}}{5} \frac{(h/2\pi)^2}{m} \frac{\hat{E}V^{-5/3}}{\hat{E}V} \quad (29.1)$$

We note two things about this expression

- Σ the presence of h reflects its quantum mechanical origin

- Σ P grows like $[\text{density}]^{5/3}$ (pressure is an energy density, and E_F grows like $[\text{density}]^{2/3}$; compare this to a classical ideal gas where P grows like $[\text{density}]^1$).

As a star burns up its nuclear fuel, at some point its temperature starts to fall, so that its thermal pressure is unable to support the pressure from gravity. Thus, the star begins to collapse, increasing the density of its core and causing the pressure of its electrons to rise, according to an ideal gas, or Eq. (29.1). For an equal density of electrons or protons, Eq. (29.1) argues that the electron pressure will be the higher by the usual factor of 2000.

Now, the pressure at the core arising from gravity is very approximately, from Lec. 24

$$P_{core} \sim \frac{4pR^2}{3} \rho^2 G$$

A slightly better calculation, *but still an approximation*, gives

$$P_{core} \sim \frac{2pR^2}{3} \rho^2 G \quad (\text{not exact}) \quad (29.2)$$

as described in Eq. (15.1) of Carroll and Ostlie. At some point during contraction, the electron pressure will exceed the gravitational pressure, and this will occur at roughly

$$\frac{2pR^2}{3} \rho^2 G = \frac{(3p^2)^{2/3}}{5} \frac{(h/2p)^2}{m_e} \frac{\hat{E}N^{5/3}}{\hat{E}V} \quad (29.3)$$

when equating (29.1) and (29.2), where Z/A is the number of electrons per nucleon. Taking the mass density to be

$$\rho = \frac{3}{4p} \frac{M}{R^3} \quad (29.4)$$

we find

$$R_{WD} \sim \frac{(18p)^{2/3}}{10} \frac{(h/2p)^2}{Gm_e M_{WD}^{1/3}} \frac{\hat{E}Z}{\hat{E}A} \sum \frac{1}{m_H} \hat{E}^{5/3} \quad (29.5)$$

Let's first of all do an example for a one-solar mass white dwarf with $Z/A = 0.5$ (for nuclei heavier than hydrogen):

$$M_{WD} = 2.0 \times 10^{30} \text{ kg}$$

$$R_{WD} \sim \frac{(18p)^{2/3}}{10} \frac{(6.63 \times 10^{-34} / 2p)^2}{6.67 \times 10^{-11} \sum 9.1 \times 10^{-31} \sum (2.0 \times 10^{30})^{1/3}} \frac{\hat{E}}{\hat{E}} \frac{1}{2 \cdot 1.67 \times 10^{-27}} \hat{E}^{5/3}$$

or

$$R_{WD} = 2.86 \times 10^6 \text{ m} = 2900 \text{ km.}$$

This is too low by a factor of two - the actual radius is close to that of the Earth.

An intriguing result can be found by rearranging Eq. (29.5) to read

$$M_{WD} R_{WD}^3 \sim \frac{(18p)^2}{10^3} \frac{(h/2p)^6}{(Gm_e)^3} \frac{\hat{E}Z}{\hat{E}A} \sum \frac{1}{m_H} \hat{E}^5 = \text{constants} \quad (29.6)$$

Derived under the assumption of constant density, this expression shows that the product of the mass and the volume ($4pR_{WD}^3/3$) is constant. In other words, as the mass of the white dwarf increases, its volume decreases.

Eq. (29.6) fails as the electrons become relativistic, because the pressure expression (29.1) is non-relativistic. Evaluating the expression exactly establishes that there is an upper bound to the density beyond which the electron pressure cannot support the pressure from gravity. In contrast, Eq. (29.6) just predicts that the star gets smaller as the mass increases. The theoretical upper limit on the mass of a white dwarf (the **Chandrasekhar** limit, after its discoverer) is 1.44 solar masses. No violations of the

Chandrasekhar limit have yet been observed.

Neutron stars

For stellar mass above the Chandrasekhar limit, further collapse will occur. Will the star now shrink without bound?

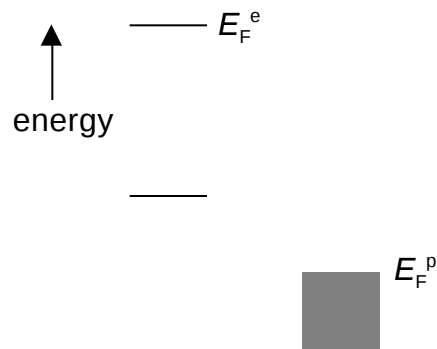
Let's assume that the star is largely electrons, protons and neutrons. The Fermi *momentum* of all particle species with the same number density will be identical, given that

$$p_F = h (3N / 8\pi V)^{1/3} \quad (29.7)$$

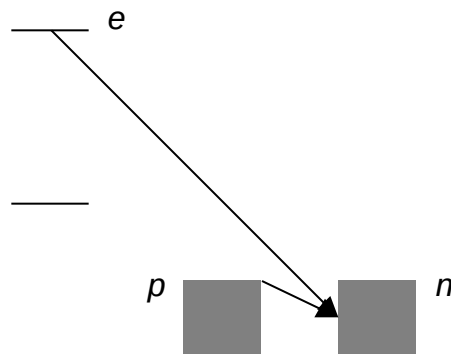
where N / V is the particle number density. However, the Fermi *energy* will vary according to particle mass, with *lighter* particles having *higher* energy:

$$E_F = p_F^2 / 2m \quad (29.8)$$

In the case of protons and electrons, the Fermi energy of the electrons will be 2000 times higher than that of the protons. Thus, the energy levels will look like (we can't do justice to a factor of 2000!)



When the difference between the Fermi energies exceeds the difference between the proton and neutron mass energies $m c^2$, an electron near E_F^e can be captured by a proton near E_F^p to create a neutron. That is



The amount of energy required for electron capture (with the release of a massless neutrino) is

$$Dm c^2 = m_n c^2 - m_p c^2 - m_e c^2$$

but

$$m_n - m_p - m_e = (1.6750 - 1.6726 - 0.00091) \times 10^{-27} = 1.5 \times 10^{-30} \text{ kg}$$

so

$$Dmc^2 = 1.5 \times 10^{-30} \cdot (3.0 \times 10^8)^2 = 1.34 \times 10^{-13} \text{ J.}$$

As $E_F^e \sim 2000 E_F^p$, let's just equate

$$E_F^e = Dmc^2 = 1.34 \times 10^{-13} \text{ J.}$$

The density corresponding to this threshold for capture can then be found from inverting Eq. (29.8) **which is a non-relativistic expression, although the electrons with E_F must be relativistic given the magnitude of Δmc^2 :**

$$E_F^e = \frac{1}{2m_e} h^2 \left(\frac{3N}{8\pi V} \right)^{2/3}$$

whence

$$\frac{(2m_e Dmc^2)^{3/2}}{h^3} \sum \frac{8\pi}{3} = \frac{N}{V}$$

and

$$\frac{N}{V} = \frac{8\pi}{3} \frac{(2 \cdot 9.11 \times 10^{-31} \times 1.34 \times 10^{-13})^{3/2}}{(6.626 \times 10^{-34})^3} = 3.5 \times 10^{36} \text{ m}^{-3}$$

This number density (of protons or electrons) is much higher than the density of our Sun, which comes in at around 10^{30} m^{-3} . Thus, the threshold for electron capture in one-solar-mass neutron star would occur at a radius about $1 / (3 \times 10^6)^{1/3} = 0.007$ that of the Sun, or

$$R_{\text{threshold}} \sim 0.007 \cdot 7 \times 10^5 = 5000 \text{ km.}$$

Caveats

- Σ we have used non-relativistic expressions for the kinetic energy of the electrons
- Σ we have ignored nuclear binding effects, taking the star's core to be hydrogen, rather than ^{56}Fe in white dwarfs
- Σ we have calculated the threshold for the formation of neutrons, rather than the equilibrium density of the star once conversion is complete (see PHYS 385).

When these, and other, approximations are properly treated, it is calculated that a fully collapsed neutron star with a mass of about 1.4 solar masses has:

- Σ a nucleon number density in the region of 10^{39} m^{-3}

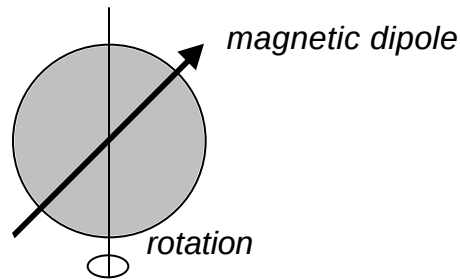
- Σ a radius of about 5 km.

This density is still much less than the density of nuclear matter at $1.1 \times 10^{14} \text{ m}^{-3}$.

Pulsars

One of the consequences of the small radius of neutron stars is their rapid spin rate. Another characteristic is the potentially large magnetic moment arising from the magnetic moments of the neutrons themselves. As a result, one expects neutron stars

to emit radiation along the axis of its magnetic dipole (which can spin around its axis of rotation, just like the Earth's magnetic axis does)



Neutron stars have been advanced as a model for *pulsars*, sources of pulsed radio emission with very short periods (whose distribution is peaked at about 0.8 s, but may range down to milliseconds in extreme cases; p. 610 of Carroll and Ostlie).

Are such short periods even possible for a collapsing star? Suppose we take our Sun, and calculate its period if it had the same mass and angular momentum $L = I\omega$, but a radius of just 5 km. The period of the Sun is 26 days for gas at the equator, somewhat longer at the poles. The moment of inertia of a sphere is

$$I = (2/5)MR^2,$$

so conservation of angular momentum would tell us

$$P_5 = P_{\text{sun}} (R_5 / R_{\text{sun}})^2 \\ = 26 \cdot 24 \cdot 3600 \cdot (5 / 7 \times 10^5)^2 = 1.1 \times 10^{-4} \text{ seconds},$$

where P is the period of the star.

Clearly, one should not expect all of a collapsing star's angular momentum to reside in its core - much of the angular momentum will be carried off by ejected material. All that this calculation establishes is that periods of seconds are quite possible for the cores of collapsing stars like our Sun.