Lecture 4 - Orbits and masses

What's important
- Kepler's laws
- determination of solar and planetary masses

Text: Carroll and Ostlie, Secs. 2.1, 2.3; review 2.2 independently

In contrast to the radii of planetary orbits, which can be determined by geometric arguments, the masses of planets and stars must be found using some aspects of Newton's law of gravitation, or its relativistic generalization. Fortunately, about half of stars are members of binary pairs, whose orbits provide several pieces of information not available from isolated stars:

orbital period orbital radius.

Our analysis is based upon these radii and periods, starting from Kepler's laws for planetary motion:
- *First law* orbits have the form of ellipses with the Sun at one focal point
- *Second law* a line connecting the planet to the Sun sweeps out equal areas in equal time
- *Third Law* $P^2 \propto a^3$, where P is the period and a is the semi-major axis.

(following Carroll and Ostlie, we use P for period and T for temperature)

We use the third law for extracting masses from planetary motion or binary stars. In second year, this law is often derived for an object moving around a fixed mass due to their gravitational attraction (see PHYS 211 notes). Here, we repeat the calculation using reduced mass.

We start with the energy of two masses m_1 and m_2, each a displacement $\mathbf{r}_1$ and $\mathbf{r}_2$ from a coordinate origin. Acting under gravity, the energy E of this pair of masses is

$$E = \tfrac{1}{2} m_1 v_1^2 + \tfrac{1}{2} m_2 v_2^2 - \frac{G m_1 m_2}{r_2 - r_1} \tag{4.1}$$

For two-body systems, it is most convenient to work with the reduced mass μ, total mass M, and relative separation $\mathbf{r}$. Placing the origin at the centre-of-mass of the system, the position vectors are

$$\mathbf{r}_1 = -\frac{m_2}{m_1 + m_2} \mathbf{r} = -\frac{\mu}{m_1} \mathbf{r} \tag{4.2}$$

and

$$\mathbf{r}_2 = \frac{m_1}{m_1 + m_2} \mathbf{r} = \frac{\mu}{m_2} \mathbf{r} \tag{4.3}$$

if

$$\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1.$$

With these definitions, and the corresponding ones for relative velocity $\mathbf{v} = d\mathbf{r}/dt$, the kinetic energy becomes

$$\tfrac{1}{2} m_1 v_1^2 + \tfrac{1}{2} m_2 v_2^2 = \tfrac{1}{2} m_1 \left(-\frac{\mu}{m_1} \mathbf{v} \right)^2 + \tfrac{1}{2} m_2 \left(\frac{\mu}{m_2} \mathbf{v} \right)^2$$

$$= \tfrac{1}{2} \left(\frac{\mu^2}{m_1} + \frac{\mu^2}{m_2} \right) \mathbf{v}^2 = \tfrac{1}{2} \mu^2 \left(\frac{1}{m_1} + \frac{1}{m_2} \right) \mathbf{v}^2 = \tfrac{1}{2} \mu \mathbf{v}^2$$

and the total energy is

$$E = \tfrac{1}{2} \mu v^2 - \frac{G\mu M}{r} \tag{4.4}$$

where we have used the replacement

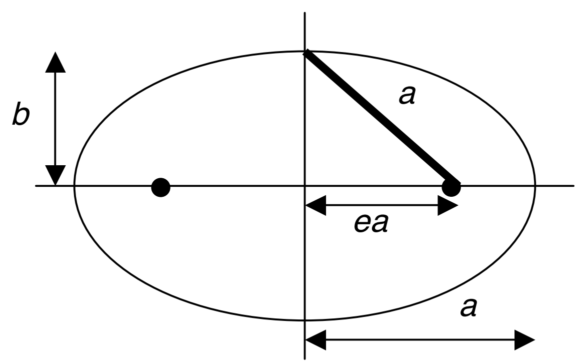
$$m_1 m_2 = \frac{m_1 m_2}{(m_1 + m_2)} (m_1 + m_2) = \mu M$$

First Law

Now, Eq. (4.4) has the same form as the fixed mass problem, with the replacement

μ = mobile mass $\quad$ M = fixed mass.

As established in the on-line lecture notes for PHYS 211, the bound orbits of this potential are elliptical in shape



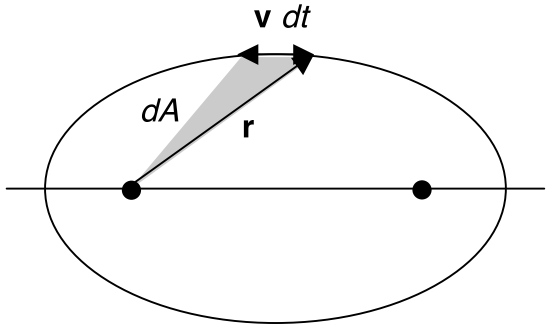
The semi-major and semi-minor axes a and b are indicated on the diagram. By definition, the sum of the distances from a point on the ellipse to the two foci is $2a$, so the heavy line in the diagram must have a length a (by symmetry). Whence, by Pythagoras:

$$a^2 = b^2 + (ea)^2 \qquad \text{or} \qquad b^2 = a^2 (1 - e^2), \tag{4.5}$$

where e is the eccentricity. A circle has $e = 0$.

Second law

Indicated by the shaded region of the diagram, the area element dA is given by the cross product rule [*area of parallelogram*] = **A x B**:



$$dA = (1/2)|\, \mathbf{r} \times \mathbf{v}dt\,|$$

or

$$\frac{dA}{dt} = \tfrac{1}{2}|\mathbf{r} \times \mathbf{v}| = \tfrac{1}{2}\frac{L}{\mu} \qquad \text{(second law)} \qquad (4.6)$$

where the second equality follows from $\mathbf{L} = \mu\mathbf{r}\mathrm{x}\mathbf{v}$.

Because angular momentum is conserved here, then dA/dt must be constant. Integrating dA/dt over the period, we have

$$A = \pi ab = \tfrac{1}{2}\frac{L}{\mu}P\,.$$

Third Law

Let's square this last expression and put P^2 on the left-hand side

$$P^2 = 4\pi^2\, a^2\, b^2\, \mu^2 \,/\, L^2,$$

then use Eq. (4.5) to rid ourselves of b:

$$P^2 = 4\pi^2\, a^4\, (1\text{-}e^2)\, \mu^2 \,/\, L^2. \qquad (4.7)$$

Our task now is to find an expression for L as a function of a and e. Consider the locations on the orbit where r has its largest and smallest values:

closest distance = $r_p = a - ea = a(1\text{-}e)$ (*e.g.* perihelion)
furthest distance = $r_a = a + ea = a(1+e)$. (*e.g.* aphelion)

The angular momentum at these two locations is the same, so

$$r_p v_p = r_a v_a$$

or

$$\frac{r_p}{r_a} = \frac{v_a}{v_p} = \frac{1-e}{1+e} \qquad (4.8)$$

Let's take this relation and substitute it into Eq. (4.4) for the energy at these two locations:

$$\tfrac{1}{2}\mu v_{\rm p}^2 - \frac{G\mu M}{a(1-e)} = \tfrac{1}{2}\mu v_{\rm p}^2\left(\frac{1-e}{1+e}\right)^2 - \frac{G\mu M}{a(1+e)}$$

which can be rearranged to give

$$\tfrac{1}{2}\mu v_{\rm p}^2\left(1-\frac{[1-e]^2}{[1+e]^2}\right) = \frac{G\mu M}{a}\left\{\frac{1}{(1-e)}-\frac{1}{(1+e)}\right\}$$

Now, the expressions in braces can be simplified to give

$$1-\frac{[1-e]^2}{[1+e]^2} = \frac{[1+e]^2-[1-e]^2}{[1+e]^2} = \frac{4e}{[1+e]^2}$$

and

$$\left\{\frac{1}{(1-e)}-\frac{1}{(1+e)}\right\} = \frac{2e}{1-e^2}$$

so that

$$\tfrac{1}{2}v_{\rm p}^2\frac{4e}{(1+e)^2} = \frac{GM}{a}\frac{2e}{\left(1-e^2\right)}$$

or

$$v_{\rm p}^2 = \frac{GM}{a}\frac{(1+e)^2}{\left(1-e^2\right)} = \frac{GM}{a}\left(\frac{1+e}{1-e}\right)$$

This expression for $v_{\rm p}^2$ can be substituted into $L = \mu r_{\rm p} v_{\rm p}$ to give

$$L^2 = \mu^2\, a^2(1-e)^2\,\frac{GM}{a}\left(\frac{1+e}{1-e}\right) = \mu^2\, aGM\left(1-e^2\right) \qquad (4.9)$$

Now we can take Eq. (4.9) back to (4.7) to obtain

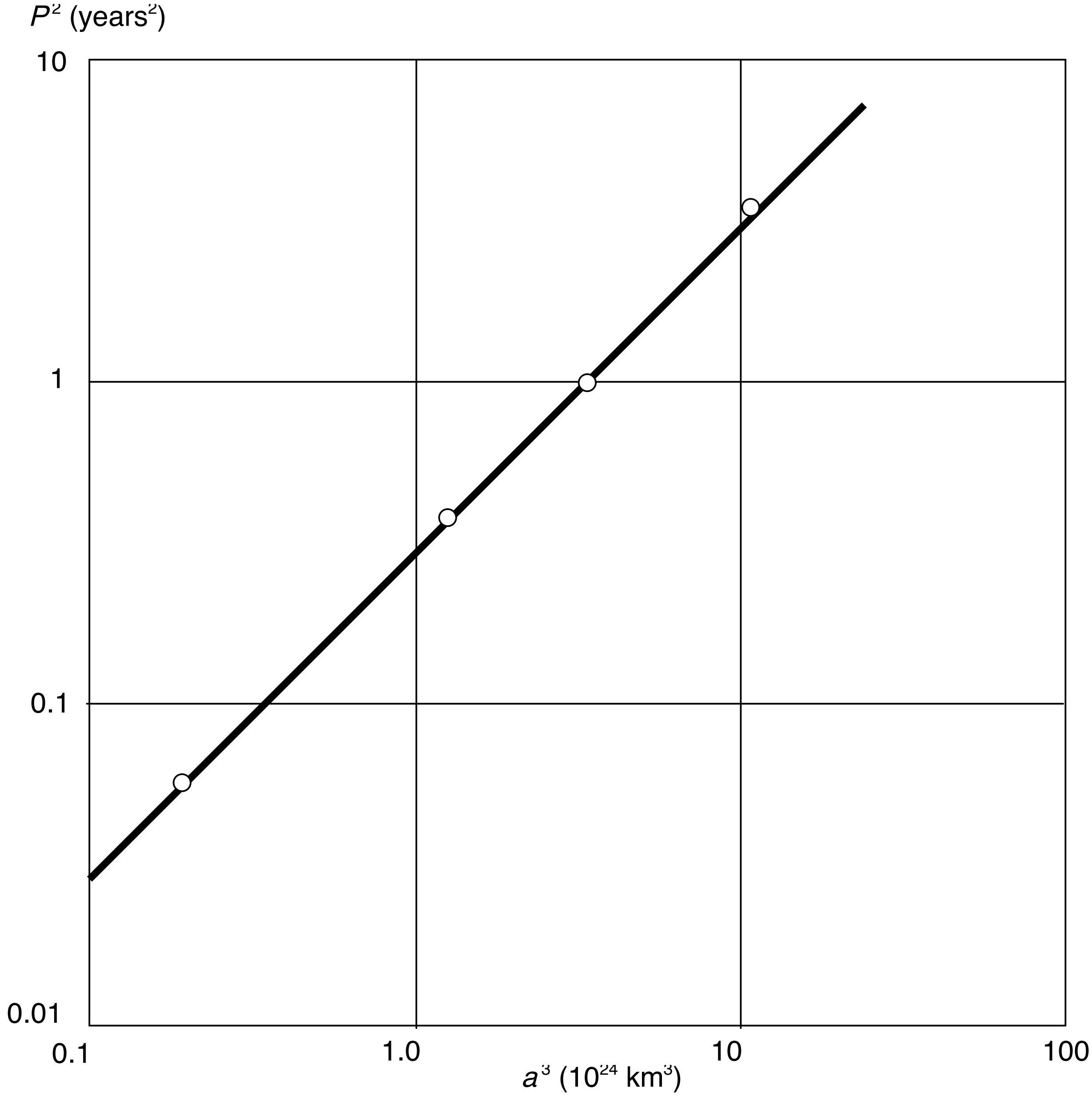
$$P^2 = \frac{4\pi^2a^4\left(1-e^2\right)\mu^2}{L^2} = \frac{4\pi^2a^4\left(1-e^2\right)\mu^2}{\mu^2 aGM\left(1-e^2\right)} = \frac{4\pi^2a^3}{GM}$$

that is

$$P^2 = \frac{4\pi^2a^3}{GM} \qquad (4.10)$$

This expression can be used to obtain accurate values for masses in a variety of circumstances. If one has several objects orbiting a common center, then a plot of P^2 against a^3 allows the mass M of the common center to be extracted from the slope (if $\mu \ll M$). Examples are the Sun, and Jupiter and binary stars.

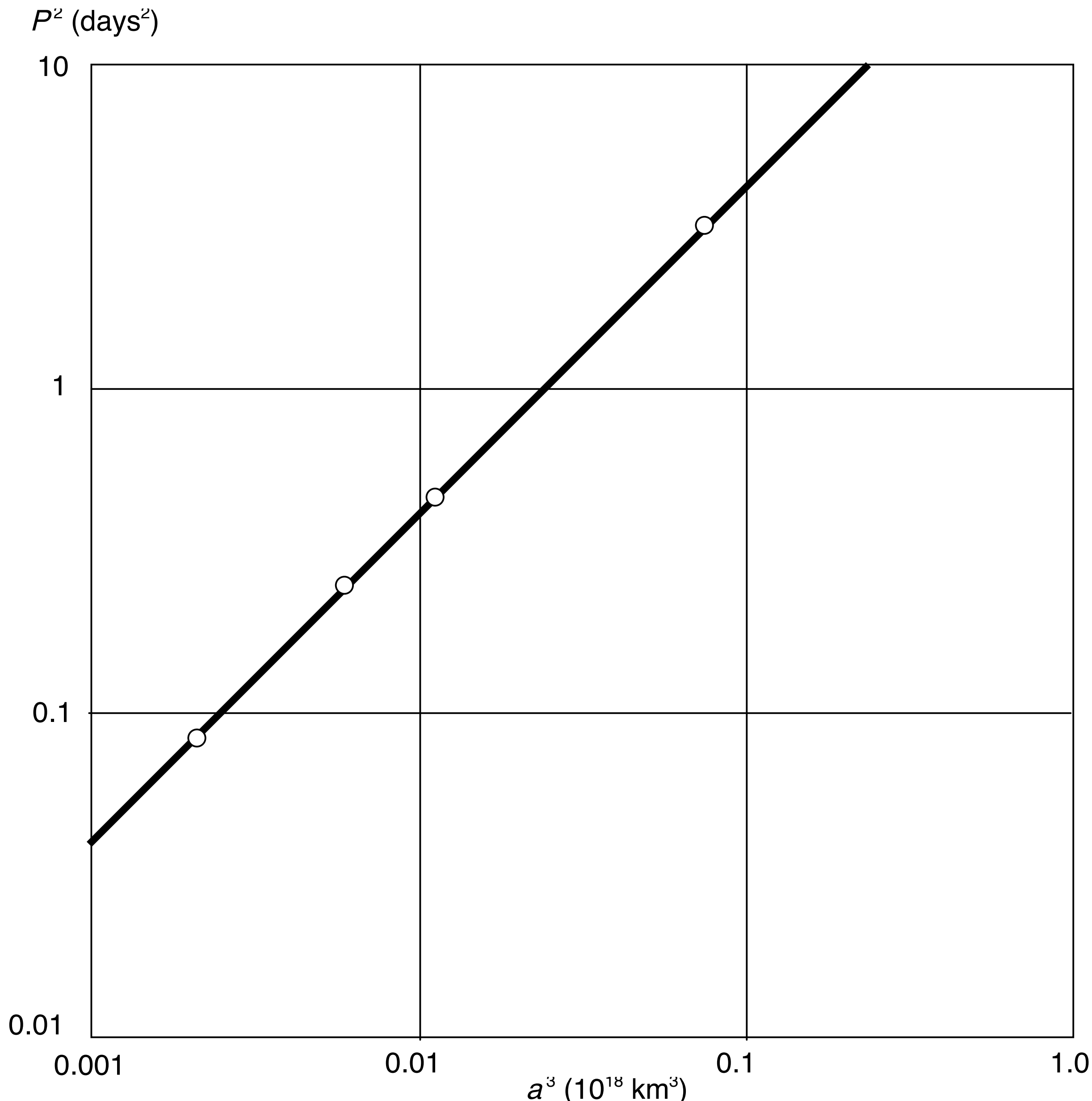
Sun



a(10^6 km)	P(years)
57.9	0.241
108.2	0.615
149.6	1.000
227.9	1.881
+more	

slope of log-log = 1
slope of linear = 2.98×10^{-18} s^2/m^3
mass = $4\pi^2$ / G•slope = 1.99×10^{29} kg (accepted = 1.99×10^{29} kg)

Jupiter



a(10^6 km)	P(days)
0.128	0.29
0.181	0.50
0.222	0.675
0.422	1.769
+ more	

slope of log-log = 1
slope of linear = 3.06×10^{-16} s^2/m^3
mass = $4\pi^2$ / G•slope = 1.93×10^{27} kg (accepted = 1.90×10^{27} kg)