Lecture 5 - The expanding universe

What's important:

- Doppler shift
- Hubble's law
- expansion against gravity

Text: Carroll and Ostlie, Secs. 5.1, 27.1
Weinberg, Chap. 2, Mathematical supplement

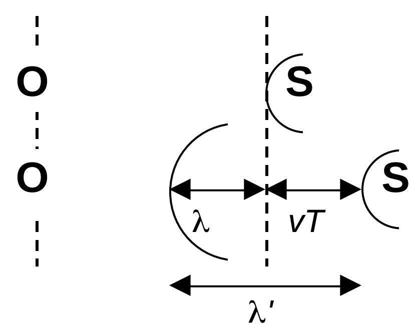
In the previous lectures, we discussed techniques for measuring the distance to, and sizes of stars, as well as their masses (if binaries). Another characteristic is their motion, both their velocity and acceleration. In this lecture, we review the Doppler shift for obtaining radial velocities (from the shift in wavelengths of light), then describe and interpret Hubble's Law for the velocities of galaxies at long distances.

Doppler effect

Consider a source of waves **S** which is moving away from an observer **O** at a speed v

first crest is emitted at $t = 0$

second crest is emitted at $t = T$



- λ is the natural wavelength of the waves emitted by the source
- in the period (of the wave) T, the source moves a distance vT
- thus, the distance between wave crests as seen by observer **O** is

$$\lambda' = \lambda + vT$$

$$(\lambda' - \lambda) / \lambda = vT/\lambda$$

- the speed c of a wave is given by $c = \lambda / T$, so

$$(\lambda' - \lambda) / \lambda = v / c. \qquad (6.1)$$

The shift in the wavelength from λ at the source to λ' at the observer is referred to as the Doppler shift of the wave. For sources moving towards the observer, the analysis is the same but the sign of v/c is reversed and $\lambda' < \lambda$. This is a non-relativistic expression, and must be modified as v/c approaches unity (see p. 108-110 of Carroll and Ostlie).

Hubble's law

The emission (and absorption) of light under some circumstances involves waves of a very precise frequency that is characteristic of the element or compound from which the light is emitted. In astronomical applications, light emitted at a specific frequency from an element is measured both on Earth and from the distant star or galaxy. Light from stars within the Milky Way, and even nearby galaxies, may be shifted to the red or blue according to their motion with respect to Earth. If the shift is due to the motion of the stars, the radial velocity can be determined from the Doppler shift.

Nearby stars within the Milky Way (a few pc) show radial velocities up to 20 km/s towards and away, with much variation (see page A-12 in the appendices of Carroll and Ostlie). For example, Sirius at 2.6 pc has a radial velocity of -7.6 km/s. The Andromeda galaxy (M31, distance of 0.77 Mpc) is approaching the Milky Way at a relative speed of 119 km/s, and the two galaxies will collide in 6.3×10^9 years.

In 1929, Edwin Hubble found that the velocity of recession V of a distant object is related to its distance from the Earth, R by

$$V = HR \qquad \text{(Hubble's law)} \qquad (6.2)$$

where H is a number called the Hubble parameter. The currently "accepted" value of the Hubble parameter is in the range 71±7 km s^{-1} Mpc^{-1}, (1 Mpc = 3.26 10^6 ly = 3.09×10^{19} km. Note that the units of H are actually $[time]^{-1}$.

Examples

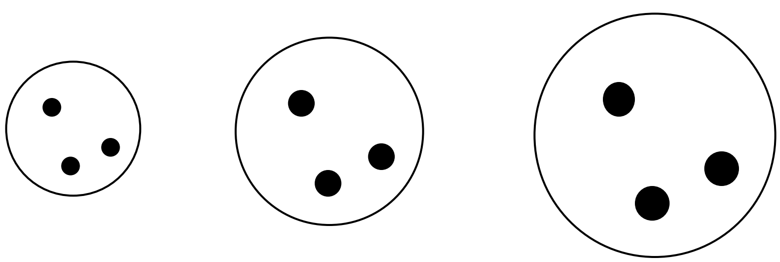
Even at a distance of 10 Mpc (Andromeda is at 0.77 Mpc) the velocity of recession from Hubble's law is not large: $V = HR \sim 70 \bullet 10 = 700$ km/s.

At a distance of a few billion light years (age of universe is of the order 10 billion years), or 1000 Mpc, the speed approaches the relativistic domain:

$V = HR \sim 70 \bullet 1000 = 70000$ km/s = 7×10^7 m/s (compared to $c = 3 \times 10^8$ m/s).

Hubble's Law and uniform expansion

Hubble's law can be illustrated by the relative motion of positions on the surface of a balloon while the balloon is being inflated



For all distances between points on the balloon's surface, the distances in one diagram are related to the distances in another diagram by a common scale factor $s(t)$ which changes with time. Let's examine a particular pair of points initially separated by a distance r_o at $t = 0$, becoming

$$r(t) = s(t)\, r_o \qquad (6.3)$$

with the passage of time *t*. The speed at which the points separate is therefore

$$v = \frac{dr}{dt} = r_o \frac{ds}{dt} = \frac{r}{s}\frac{ds}{dt} \qquad \text{after replacing } r_o \text{ with } r(t)/s(t)$$

This expression no longer contains a reference to the original position, only

$$v = r\frac{\dot{s}}{s}$$

from which the Hubble parameter can be identified as

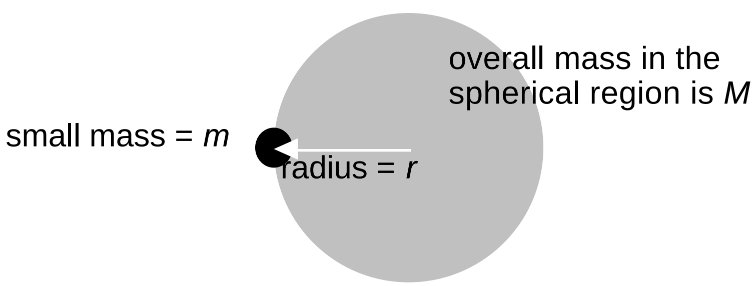
$$H = \frac{\dot{s}}{s}.$$

In other words, Hubble's law applies to the motion of particles on the surface of a balloon, or to any system that is expanding according to a scale transformation like Eq. (6.3), and applies to all points in the system. The system has no "central" position.

Expansion against gravity

Let's find the time-dependence of *H* of a mass expanding against the gravitational pull of its surroundings. Consider a small mass *m* a distance *r* away from an arbitrarily chosen origin, embedded in a uniform matter distribution of density ρ. If the region contains a total mass *M*, the density is

$$\rho = [density] = M\, /\, [(4\pi/3)\, r^3]$$



Newton's law of gravity tells us the acceleration experienced by the mass

$$ma = F = -GMm\, /r^2$$

or

$$\frac{d\dot{r}}{dt} = -\frac{GM}{r^2}$$

Now, the left-hand side of this equation can be reworked using

$$\frac{d\dot{r}^2}{dt} = \frac{d\dot{r}}{dt}\cdot\frac{d\dot{r}^2}{d\dot{r}} = \frac{d\dot{r}}{dt}\cdot 2\dot{r}$$

so that

$$\frac{1}{2\dot{r}}\frac{d\,\dot{r}^2}{dt} = -\frac{GM}{r^2}$$

or

$$\frac{d\,\dot{r}^2}{dt} = -2GM\,\frac{\dot{r}}{r^2}$$

This may not look like an improvement, but we can now simplify the right hand side:

$$\frac{d}{dt}\left(\frac{1}{r}\right) = \frac{0-\dot{r}}{r^2} = -\frac{\dot{r}}{r^2}$$

so

$$\frac{d\,\dot{r}^2}{dt} = 2GM\,\frac{d}{dt}\left(\frac{1}{r}\right)$$

Now, we simply integrate out the *d /dt* to obtain

$$\dot{r}^2 = \frac{2GM}{r} + K \qquad (6.4)$$

where K is an integration constant.

Our next task is to remove the physical $r(t)$ in favour of the scale factors $s(t)$. First, we write

$$\frac{\dot{r}^2}{r^2} = \frac{2GM}{r^3} + \frac{K}{r^2}$$

and then use the ratios

$$\frac{\dot{r}}{r} = \frac{\dot{s}}{s}$$

so

$$\left(\frac{\dot{s}}{s}\right)^2 = \frac{2GM}{r^3} + \frac{K}{r^2}$$

Now, the usual miracle occurs. The enclosed mass M is proportional to r^3 via

$$M = \frac{4\pi}{3} r^3 \rho$$

leaving us with

$$\left(\frac{\dot{s}}{s}\right)^2 = \frac{8\pi G\rho}{3} + \frac{K}{r^2}$$

The integration constant is fixed by the boundary conditions. If the kinetic and potential energy of the test mass are balanced, then

$$\frac{1}{2} m\dot{r}^2 = \frac{GmM}{r} \qquad (6.5)$$

and

$K = 0$ (kinetic energy = potential energy)

by substituting Eq. (6.5) into Eq. (6.4)

Replacing the left hand side of the equation with H^2 and taking the square root, we have the important

$$H = \left(\frac{8\pi G\rho}{3}\right)^{1/2} . \quad (6.6)$$

Although we have established this using Newtonian mechanics, the result applies relativistically as well. All we need to do is replace the mass density ρ with the energy density U and divide by c^2:

$$H = \left(\frac{8\pi G U}{3c^2}\right)^{1/2} . \quad (6.7)$$

Both forms, (6.6) and (6.7), show that H vanishes at infinite dilution: *i.e.*, the universe slows to a stop at infinite time. This is what we expect from our boundary condition equating the kinetic and potential energies. Of course, the universe may choose to do otherwise:

K.E. > *P.E.* universe expands forever
K.E. < *P.E.* universe contracts again at finite time.

Example

What is the critical density according to Eq. (6.6) if H = 70 km / s•Mpc? How does this value compare with the observed density of baryonic matter?

Start with 1 Mpc = 3.09 x 10^{19} km, so that

$$H = 70 / 3.09 \times 10^{19} = 2.3 \times 10^{-18}\ \mathrm{s}^{-1}.$$

Then

$$H = \left(\frac{8\pi G\rho}{3}\right)^{1/2}$$

gives

$$\rho_{crit} = 3H^2 / 8\pi G$$
$$= 3\,(2.3 \times 10^{-18})^2 / (8\pi \cdot 6.67 \times 10^{-11}) = 9.4 \times 10^{-27}\ \mathrm{kg/m^3}.$$

The observed value of the density, based upon baryonic matter, is almost a hundred times less:

$$\rho_{baryon} = 2\text{-}5 \times 10^{-28}\ \mathrm{kg/m^3},$$

or

$$\rho_{baryon} / \rho_{crit} = 2\text{-}5\ \%.$$

Inclusion of dark matter, as obtained from galactic rotation, raises the matter estimate by about a factor of ten, to

$$\rho_{matter} / \rho_{crit} \sim 30\%.$$

Other contributions are made by radiation *etc.*, so that the total ratio approaches unity.