Classical Mechanics.

Before launching into the formalism of quantum mechanics, we wish to review several different approaches to the description of motion. Three different approaches will be emphasized, the last of which yields the easiest generalization. Of course, for a particular system, each approach yields the same answer; just different aspects of the problem are emphasized.

Newtonian Mechanics – forces, positions (and velocities)

Lagrangian approach – q, $\dot{q}$, Lagrangian

Hamiltonian approach – q, p, total energy.

Newtonian mechanics

This is, in some sense, the least formal. One deals directly with positions and forces:

$\vec{r}(t)$

$\vec{F}(\vec{r}, t)$ ← here we will assume that this is derivable from a potential

$$\vec{F} = -\vec{\nabla} V \quad \text{[no dissipation]}.$$

$$\Rightarrow \vec{\nabla} \times \vec{F} = 0 \quad \text{(irrotational)}.$$

Start with $\vec{r}(t=0)$ and $\left.\frac{d}{dt}\vec{r}\right\|_{t=0} = \vec{v}(t=0)$	Then integrate using $\frac{d^2}{dt^2}\vec{r} = \frac{1}{m}\vec{F}$.

Two quantities are of importance here,

① $E = \frac{1}{2}mv^2 + V(\vec{r}, t)$

② $\vec{L} = m\vec{r} \times \vec{v}$

Here, we will ignore the momentum since at this level momentum is just the velocity times a scalar so we will just use velocity. We can show that these quantities are constants of motion in certain circumstances:

② Central potential

$$\frac{d}{dt}\vec{L} = \frac{d}{dt} m\vec{r}\times\vec{v} = m\left[\left(\frac{d\vec{r}}{dt}\right)\times\vec{v} + \vec{r}\times\frac{d\vec{v}}{dt}\right]$$

$$= m\left[\underbrace{\vec{v}\times\vec{v}}_{=0} + \vec{r}\times\vec{a}\right]$$

If the potential is central $V(\vec{r}) = V(r)$

$\Rightarrow \vec{F} = -\vec{\nabla} V(r)$ points along $\vec{r}$.

$\Rightarrow \vec{r} \times \underset{\text{some function}}{S}\,\vec{r} = 0$

$\therefore \frac{d}{dt}\vec{L} = 0$ for central potentials.

Similarly, for any irrotational force with a time independent potential

$$\frac{dE}{dt} = \frac{d}{dt}\left[\frac{1}{2}mv^2 + V(\vec{r})\right]$$

$$= \frac{1}{2}m \cdot 2\vec{v}\,\frac{d\vec{v}}{dt} + \frac{d}{dt}V(\vec{r})$$

$$= m\vec{v}\cdot\underbrace{\frac{d\vec{v}}{dt}}_{\vec{a}} + \underbrace{\frac{d\vec{r}}{dt}}_{\vec{v}}\cdot\vec{\nabla}V(r)$$

$$= \vec{v}\cdot(m\vec{a} + \vec{\nabla}V) = 0 \text{ by } \vec{F} = m\vec{a} = -\vec{\nabla}V.$$

$$\frac{d}{dx}V(r) = \frac{dr}{dx}\frac{dV}{dr}$$

$$= \frac{d(x^2+y^2+z^2)^{1/2}}{dx}\frac{dV}{dr}$$

$$= \frac{1}{2}\cdot\frac{1}{r}\cdot 2x\,\frac{dV}{dr}$$

$$= \frac{x}{r}\frac{dV}{dr}$$

forms a unit vector

Lagrange's Equations

Newtonian mechanics emphasizes the importance of positions and forces; the velocities (and their change) are determined by the force law, while energy is a derived quantity. A different approach which makes the positions and velocities somewhat more symmetrical, and places greater emphasis on the energy, is the Lagrangian approach:

$\mathcal{L} \equiv T - V$ where T is the kinetic energy.

T & V are functions of generalized coordinates and velocities. q_i, $\dot{q}_i$

The equations of motion are then obtained from the Euler-Lagrange equations (from Symon:)

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \frac{\partial \mathcal{L}}{\partial q_i}$$ ← These are N second order equations.

Proof: Long and needs calculus of variations. So ignore.

Find connection with Newtonian mechanics.

<u>Example I</u>: In cartesian coordinates:

$$\mathcal{L} = \frac{1}{2}m\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right) - V(x,y,z,t)$$

Applying the E-L equation gives (say to x-coordinate)

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = \frac{\partial \mathcal{L}}{\partial x} \Rightarrow \frac{d}{dt}\left(\frac{1}{2}m\cdot 2\dot{x}\right) = -\frac{\partial V}{\partial x}$$

$$\Rightarrow m\ddot{x} = -\frac{\partial V}{\partial x} = F_x \quad \text{if } F_x \equiv -\frac{\partial V}{\partial x}.$$

Thus, we regain Newton's Laws immediately.

Further, if we define a momentum p_i associated with coordinate q_i via

$$p_i \equiv \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$$

Then $$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \frac{\partial \mathcal{L}}{\partial q_i} \Rightarrow \dot{p}_i = \frac{\partial \mathcal{L}}{\partial q_i} = \underline{\underline{\frac{\partial T}{\partial q_i}}} - \underline{\underline{\frac{\partial V}{\partial q_i}}}$$

Looks like normal force

extra term. For cartesian coord., this vanishes.

<u>Example II</u> In plane polar coordinates (2-D): (r,φ)

$$\dot{x}^2+\dot{y}^2 = \left[\frac{d}{dt}(r\cos\varphi)\right]^2 + \left[\frac{d}{dt}(r\sin\varphi)\right]^2$$

$$= \left[\dot{r}\cos\varphi - r\dot{\varphi}\sin\varphi\right]^2 + \left[\dot{r}\sin\varphi + r\dot{\varphi}\cos\varphi\right]^2$$

3-D polar coords:

$$\dot{x}^2+\dot{y}^2+\dot{z}^2 = \left[\frac{d}{dt}(r\sin\theta\cos\varphi)\right]^2 + \left[\frac{d}{dt}(r\sin\theta\sin\varphi)\right]^2 + \left[\frac{d}{dt}(r\cos\theta)\right]^2$$

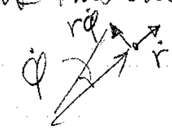
$$= \left[\dot{r}\sin\theta\cos\varphi + r\dot{\theta}\cos\theta\cos\varphi - r\dot{\varphi}\sin\theta\sin\varphi\right]^2 + \left[\dot{r}\sin\theta\sin\varphi + r\dot{\theta}\cos\theta\sin\varphi + r\dot{\varphi}\sin\theta\cos\varphi\right]^2 + \left[\dot{r}\cos\theta - r\dot{\theta}\sin\theta\right]^2$$

$$\begin{aligned}
= {} & \dot{r}^2\sin^2\theta\cos^2\varphi + r^2\dot{\theta}^2\cos^2\theta\cos^2\varphi + r^2\dot{\varphi}^2\sin^2\theta\sin^2\varphi \\
& + 2r\dot{r}\dot{\theta}\sin\theta\cos\theta\cos^2\varphi - 2r\dot{r}\dot{\varphi}\sin^2\theta\sin\varphi\cos\varphi \\
& - 2r^2\dot{\theta}\dot{\varphi}\sin\theta\cos\theta\sin\varphi\cos\varphi \\
& + \dot{r}^2\sin^2\theta\sin^2\varphi + r^2\dot{\theta}^2\cos^2\theta\sin^2\varphi + r^2\dot{\varphi}^2\sin^2\theta\cos^2\varphi \\
& + 2r\dot{r}\dot{\theta}\sin\theta\cos\theta\sin^2\varphi + 2r\dot{r}\dot{\varphi}\sin^2\theta\sin\varphi\cos\varphi \\
& + 2r^2\dot{\theta}\dot{\varphi}\sin\theta\cos\theta\sin\varphi\cos\varphi \\
& + \dot{r}^2\cos^2\theta + r^2\dot{\theta}^2\sin^2\theta - 2r\dot{r}\dot{\theta}\sin\theta\cos\theta
\end{aligned}$$

$$= \dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\varphi}^2\sin^2\theta$$

2-5.

[could also work this out by looking at motion at arc.



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$$= \dot{r}^2 + r^2\dot{\varphi}^2$$

$$\Rightarrow \quad \mathcal{L} = \tfrac{1}{2}m\left(\dot{r}^2 + r^2\dot{\varphi}^2\right) - V(r,\varphi,t).$$

Now, the expressions for the momentum are a little different:

$$\underbrace{\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{r}}\right)}_{\downarrow} = \frac{\partial\mathcal{L}}{\partial r} \rightarrow 0 + \frac{\partial\frac{m}{2}r^2\dot{\varphi}^2}{\partial r} - \frac{\partial V}{\partial r} = mr\dot{\varphi}^2 - \frac{\partial V}{\partial r}$$

$$p_r \equiv \frac{\partial\mathcal{L}}{\partial\dot{r}} = \tfrac{1}{2}m\cdot 2\dot{r} = m\dot{r}$$

$$\therefore \quad \frac{d}{dt}p_r = mr\dot{\varphi}^2 - \frac{\partial V}{\partial r}$$

This is the motion from the centrifugal force — This is the usual force term.

Similarly, the momentum corresponding to φ reads.

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\varphi}}\right) = \frac{\partial\mathcal{L}}{\partial\varphi} \cancel{\hphantom{x}} - \frac{\partial V}{\partial\varphi}$$

$$p_\varphi = \tfrac{1}{2}mr^2\,2\dot{\varphi}$$
$$= mr^2\dot{\varphi}$$

$$\text{or,} \quad \frac{dp_\varphi}{dt} = \frac{d}{dt}\left(mr^2\dot{\varphi}\right) = -\frac{\partial V}{\partial\varphi}$$

This is the torque.

For a central potential, $V = V(r,t)$; no φ. $\therefore \frac{dp_\varphi}{dt} = -\frac{\partial V}{\partial\varphi} = 0$

All of this can be generalized to velocity dependent potentials as well: $V = V(q_i, \dot{q}_i, t)$. e.g. $q\vec{v} \times \vec{B}$ for a charge in a magnetic field.

Hamilton's Equations

Lagrange's equations emphasized q_i & $\dot{q}_i$, and the energy combination $T-V$. A different approach is to emphasize q_i and its canonically conjugate momentum

$$p_i \equiv \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$$ ← use this to find a relation between p_i and $\dot{q}_i$.

Then, the Hamiltonian is defined by

$$\mathcal{H} \equiv \sum_{i=1}^{3} p_i \dot{q}_i - \mathcal{L}$$ ← use relation between p_i and $\dot{q}_i$ to eliminate $\dot{q}_i$

Express this in terms of p_i, q_i

Using the Lagrange equations, we can get Hamilton's equations:

$$d\mathcal{H} = \sum_i p_i \, d\dot{q}_i + \sum_i \dot{q}_i \, dp_i - \sum_i \underbrace{\frac{\partial \mathcal{L}}{\partial q_i}}_{\dot{p}_i} dq_i - \sum_i \underbrace{\frac{\partial \mathcal{L}}{\partial \dot{q}_i}}_{\text{This is } p_i} d\dot{q}_i - \frac{\partial \mathcal{L}}{\partial t} dt.$$

$$= \sum_i \dot{q}_i \, dp_i - \sum \dot{p}_i \, dq_i - \frac{\partial \mathcal{L}}{\partial t} dt$$

$$\Rightarrow \quad \frac{\partial \mathcal{H}}{\partial p_i} = \dot{q}_i \qquad \frac{\partial \mathcal{H}}{\partial q_i} = -\dot{p}_i \qquad \frac{\partial \mathcal{H}}{\partial t} = -\frac{\partial \mathcal{L}}{\partial t}.$$

These are Hamilton's Equations.

Hamilton's Eqn. | Lagrange's Eqn.

1 [$2 \times N$ equations | N equations

Where N is the # of degrees of freedom]

2 1st order | 2nd order

$(\dot{\ }) = \frac{\partial \mathcal{H}}{\partial \ }$ | $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) = \frac{\partial \mathcal{L}}{\partial q_i}$

Good for both $\dot{q}, \dot{p}$ [integrate them independently in a sense]. | i.e. solve for $\dot{q}_i$, then use to get q_i.

What is the Hamiltonian? For velocity-independent potentials,

$$\mathcal{H} = \sum p_i \dot{q}_i - \mathcal{L}$$

$$\mathcal{L} = T - V$$

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \frac{\partial T}{\partial \dot{q}_i}$$

For a stationary coordinate system [$q = q(x, y \ldots)$ but not t] then

$$T = \frac{1}{2} \sum_{i=1}^{N} m_i \sum_{k,\ell} \left(\frac{\partial x_i}{\partial q_k} \frac{\partial x_i}{\partial q_\ell} + \frac{\partial y_i}{\partial q_k} \frac{\partial y_i}{\partial q_\ell} + \frac{\partial z_i}{\partial q_k} \frac{\partial z_i}{\partial q_\ell} \right) \dot{q}_k \dot{q}_\ell$$

cartesians

$$\Rightarrow \sum_i \frac{\partial T}{\partial \dot{q}_i} \dot{q}_i = 2T \qquad \Rightarrow \mathcal{H} = 2T - (T - V) = T + V.$$

[for a moving coordinate system things are a little more complicated]

Poisson Brackets

Suppose we have some general function $G(q_i, p_i, t)$. Then

$$\frac{dG}{dt} = \sum_i \frac{\partial G}{\partial q_i}\underbrace{\frac{\partial q_i}{\partial t}}_{= \dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i}} + \sum_i \frac{\partial G}{\partial p_i}\underbrace{\frac{\partial p_i}{\partial t}}_{= \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i}} + \frac{\partial G}{\partial t}$$

$$\therefore \frac{dG}{dt} = \underbrace{\sum_i \left(\frac{\partial G}{\partial q_i}\frac{\partial \mathcal{H}}{\partial p_i} - \frac{\partial G}{\partial p_i}\frac{\partial \mathcal{H}}{\partial q_i}\right)} + \frac{\partial G}{\partial t}.$$

This quantity appears frequently enough to be given a name: Poisson bracket.

$$\{A, B\} = \sum_i \frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial B}{\partial q_i}\frac{\partial A}{\partial p_i}$$

So that $$\frac{dG}{dt} = \frac{\partial G}{\partial t} + \{G, \mathcal{H}\} \qquad (1)$$

Note that if G contains no explicit time dependence, $\frac{\partial G}{\partial t} = 0$, then conservation of G implies

$$\{G, \mathcal{H}\} = 0.$$

Since the functional form of Eq. (1) looks like the same kind of expression as was used for the derivation of Hamilton's eqns, it should be expected that they can

A few interesting relationships are:

$$\{q_k, q_\ell\} = \sum_i \left(\frac{\partial q_k}{\partial q_i} \frac{\partial q_\ell}{\partial p_i} - \frac{\partial q_k}{\partial q_i} \frac{\partial q_\ell}{\partial p_i} \right) = 0 \qquad \text{since } \frac{\partial q}{\partial p} = 0$$

same with $\{p_k, p_\ell\}$.

And -

$$\{q_k, p_\ell\} = \sum_i \left(\frac{\partial q_k}{\partial q_i} \frac{\partial p_\ell}{\partial p_i} - \frac{\partial p_\ell}{\partial q_i} \frac{\partial q_k}{\partial p_i} \right)$$

$= 1$ if $k = \ell$ [only $i = k = \ell$ survives, and second term vanishes].

$= 0$ if $k \neq \ell$

$= \delta_{k\ell}$ (Kronecker delta).

For angular momenta, it can be shown (homework) that

$\{L_i, L_j\} = L_k$ if cyclic. using $L = r \times p$

$\{L_i, L^2\} = 0.$