3-Dimensional Motion.

Much of this material was covered in P385 or is a simple extension of the 1-D problems just examined. We will give the results only a cursory overview, then look at a particular problem, Larmor precession, in more detail.

1. Free particle motion:

This is a straightforward extension of the 1-D case. For plane waves, one uses the usual separation of variables approach:

$$\psi(x,y,z) = X(x)\,Y(y)\,Z(z)$$

$$-\frac{\hbar^2}{2m}\nabla^2\psi = E\psi = \frac{\hbar^2 k^2}{2m}\psi$$

$$\Rightarrow \frac{1}{X}\frac{d^2}{dx^2}X + \frac{1}{Y}\frac{d^2 Y}{dy^2} + \frac{1}{Z}\frac{d^2 Z}{dz^2} = -k^2 = -(k_x^2 + k_y^2 + k_z^2)$$

or

$$\psi \propto e^{ik_x x}\, e^{ik_y y}\, e^{ik_z z} = e^{i\vec{k}\cdot\vec{r}}.$$

Note: spreading problem is now in 3 dimensions.

2. Particle in a box: (infinite potential)

For a "square well" potential (i.e. not spherical) the SE separates out as in the free particle case.

$V = 0 \quad -\frac{a}{2} \le x \le \frac{a}{2}$

$-\frac{a}{2} \le y \le \frac{a}{2}$

$V = \infty$ otherwise etc.

As usual, the boundary conditions now restrict the form of the plane wave solutions and impose the quantization condition

$$k_x = \frac{n_x \pi}{a} \qquad k_y = \frac{n_y \pi}{a} \qquad k_z = \frac{n_z \pi}{a} \quad \text{etc.}$$

The energy of each state can be expressed as

$$E(n_x, n_y, n_z) = \frac{\hbar^2 \pi^2}{2ma^2}\left(n_x^2 + n_y^2 + n_z^2\right)$$

3. Anisotropic Oscillator

The anisotropic oscillator potential has the form

$$\frac{1}{2}K_x x^2 + \frac{1}{2}K_y y^2 + \frac{1}{2}K_z z^2 .$$

Using $\omega^2 = \frac{K}{m}$ to eliminate the K:

$$-\frac{\hbar^2}{2m}\nabla^2\psi + \frac{1}{2}m\left(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2\right)\psi = \epsilon\psi .$$

As before, this factorizes into 3 separate, equivalent, 1-D problems.

$$-\frac{\hbar^2}{2m}\frac{d^2 X(x)}{dx^2} + \frac{1}{2}m\omega_x^2 x^2 X(x) = \epsilon_x X(x) \qquad \Leftrightarrow y \leftrightarrow z .$$

The 1-D solution was obtained in the following way:

First, change to dimensionless variables $\xi = \sqrt{\alpha}\, x$

$$\Rightarrow \quad \frac{d^2\psi}{d\xi^2} + \left(\frac{\lambda}{\alpha} - \xi^2\right)\psi = 0 \qquad \text{where } \lambda = \frac{2mE}{\hbar^2}.$$

Then, write $\psi = G(\xi)H(\xi)$ and take large ξ limit

↑ asymptotic solution at large ξ

$$\frac{d^2G}{d\xi^2} - \xi^2 G = 0 \quad \Rightarrow \quad G(\xi) = e^{-\xi^2/2} \qquad (e^{+\xi^2/2} \text{ diverges}).$$

Next, substitute for $G(\xi)$ and get an equation for $H(\xi)$:

$$\frac{d^2H(\xi)}{d\xi^2} - 2\xi\frac{dH(\xi)}{d\xi} + \left(\frac{\lambda}{\alpha} - 1\right)H(\xi) = 0.$$

We find the solution by series expansion of $H(\xi) = \sum_{i=0}^{\infty} a_i \xi^i$

The equation generates a recursion relation

$$a_{n+2} = -\left[\frac{(\lambda/\alpha) - (1+2n)}{(n+1)(n+2)}\right]a_n$$

The series can be made finite for each value of i by demanding

$$\frac{\lambda}{\alpha} - (1+2n) = 0$$

$$\frac{\lambda}{\alpha} = \frac{2mE}{\hbar^2} \Big/ \frac{m\omega}{\hbar} = \frac{2E}{\omega\hbar} \qquad \Rightarrow \qquad E = \omega\hbar\left(n + \tfrac{1}{2}\right) \qquad n = 0, 1, 2 \ldots$$

The general form of the Hermite polynomials can be obtained by putting a_1 or $a_0 \neq 0$, solving for all other

$$\psi_n = N_n e^{-\xi^2/2} H_n(\xi)$$

where $N_n = \left(\frac{\alpha}{\pi}\right)^{1/4}\left(\frac{1}{2^n n!}\right)^{1/2}$

$$H_0(\xi) = 1$$
$$H_1(\xi) = 2\xi$$
$$H_2(\xi) = -2 + 4\xi^2$$
$$H_3(\xi) = -12\xi + 8\xi^3 .$$

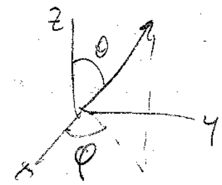
In the 3-D case, the energy becomes

$$E = \omega_x\hbar\,(n_x + 1/2) + \omega_y\hbar\,(n_y + 1/2) + \omega_z\hbar\,(n_z + 1/2)$$

Central Potentials

Let's now move away from cartesian-like systems to potentials with spherical symmetry. Spherical polar coordinates are a more natural choice for these problems than are cartesian coordinates:

$$x = r\sin\theta\cos\varphi$$
$$y = r\sin\theta\sin\varphi$$
$$z = r\cos\theta$$



$$\Rightarrow \nabla^2 = \frac{1}{r^2\sin\theta}\left(\frac{\partial}{\partial r}\left(r^2\sin\theta\frac{\partial}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{\partial}{\partial\varphi}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}\right)\right)$$

The usual separation of variables approach leads to three equations:

$$\psi = R(r)\,\Theta(\theta)\,\Phi(\varphi)$$ 5-5.

$\Theta(\theta)\Phi(\varphi) \to Y_{\ell m}(\theta,\varphi)$

$$\left\{ -\frac{\hbar^2}{2m}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + V(r) + \frac{\hbar^2 \ell(\ell+1)}{2mr^2} \right\} R = E R$$

$$\frac{d^2}{d\varphi^2}\Phi = -m^2\Phi$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \left(\ell(\ell+1) - \frac{m^2}{\sin^2\theta}\right)\Theta = 0.$$

↳ These two equations arise when $V(\vec{r}) = V(r)$ and are associated with the angular momentum states.

Angular Momentum

Before discussing particular potentials, we recap some properties of the angular momentum states: $Y_{\ell m}$.

$$\hat{L}^2 \equiv \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]$$

$$\hat{L}_z = -i\hbar\frac{\partial}{\partial\varphi}$$

$$\hat{L}^2 Y_{\ell m} = \ell(\ell+1)\hbar^2 Y_{\ell m}$$

$$\hat{L}_z Y_{\ell m} = m\hbar Y_{\ell m}.$$

(these follow directly from the form of the D.E.)

The functional form of Θ and Φ can be obtained from their Diff Equ's:

$$\Phi = 1 e^{\pm im\varphi} \qquad m = 0\ \pm1\ \pm2\ \ldots \text{/ by continuity.}$$

The Θ equation is solved by substitution

$$u = \cos\theta$$

$$\Rightarrow \frac{d}{du}\left[(1-u^2)\frac{d}{du}\Theta_{lm}\right] + \left(l(l+1) - \frac{m^2}{1-u^2}\right)\Theta_{lm} = 0$$

Substituting

$$\Theta_{lm} = (1-u^2)^{\frac{|m|}{2}} \frac{d^{|m|}}{du^{|m|}} P_l(u) \equiv P_l^m(u)$$

$$\Rightarrow P_l(u) \text{ satisfies } \frac{d}{du}\left[(1-u^2)\frac{dP_l}{du}\right] + l(l+1)P_l(u) = 0.$$

Doing the usual series expansion for P_l, $\left(\sum_{i=0}^{\infty} a_i u^i\right)$ one finds the recursion relation

$$a_{i+2} = \frac{i(i+1) - l(l+1)}{(i+1)(i+2)} a_i$$

which vanishes when $i = l$. So we obtain a sequence of polynomials [Legendre polynomials] which can be summarized in an unnormalized form as:

$P_0(u) = 1$

$P_1(u) = u$

$P_2(u) = \frac{1}{2}(3u^2 - 1)$

$P_3(u) = \frac{1}{2}(5u^3 - 3u)$ ---

The orthonormal set is then

$$\Rightarrow Y_{lm}(\theta,\varphi) = \left[\frac{2l+1}{4\pi}\frac{(l-|m|)!}{(l+|m|)!}\right]^{1/2} P_l^m(\cos\theta)\, e^{im\varphi}$$

vanish when $|m| \geq l$

$\Rightarrow m = 0, \pm 1 \cdots \pm (l-1)$

Raising and Lowering operators:

Since $[\hat{L}_i, \hat{L}_j] = i\hbar\,\epsilon_{ijk}\hat{L}_k$ $\quad \epsilon_{ijk} = +1$ $\;i,j,k$ cyclic perm.
$\qquad = -1$ otherwise
$\qquad = 0$ $\;i \neq j \neq k$ untrue.

Then only one component is conserved. We select that component as L_z.

Now, with $\hat{L}_x$ and $\hat{L}_y$ we can construct two operators L^+ & L^-: non hermitian

$$\hat{L}^\pm = \hat{L}_x \pm i\hat{L}_y$$

What is the effect of $\hat{L}^\pm$ on a state $|l\,m\rangle$?

$$\hat{L}_z\{\hat{L}^\pm|l,m\rangle\} = \{\hat{L}_z\hat{L}_x \pm i\hat{L}_z\hat{L}_y\}|l\,m\rangle$$

Solve from
$[\hat{L}_x, \hat{L}_z] = i\hbar(-1)\hat{L}_y$
$\hat{L}_x\hat{L}_z - \hat{L}_z\hat{L}_x = -i\hbar\hat{L}_y$
or $\hat{L}_z\hat{L}_x = \hat{L}_x\hat{L}_z + i\hbar\hat{L}_y$

$[\hat{L}_y, \hat{L}_z] = i\hbar(+1)\hat{L}_x$
$\hat{L}_y\hat{L}_z - \hat{L}_z\hat{L}_y = i\hbar\hat{L}_x$
$\Rightarrow \hat{L}_z\hat{L}_y = \hat{L}_y\hat{L}_z - i\hbar\hat{L}_x$

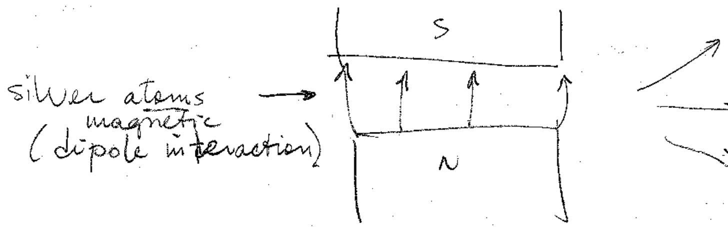
$$\Rightarrow \hat{L}_z\hat{L}_x \pm i\hat{L}_z\hat{L}_y = \hat{L}_x\hat{L}_z + i\hbar\hat{L}_y \pm i[\hat{L}_y\hat{L}_z - i\hbar\hat{L}_x]$$
$$= (\hat{L}_x \pm i\hat{L}_y)\hat{L}_z + i\hbar\hat{L}_y \pm \hbar\hat{L}_x$$
$$= \hat{L}^\pm\hat{L}_z \pm \hbar(\hat{L}_x \pm i\hat{L}_y)$$

$$\Rightarrow \hat{L}_z\{\hat{L}^\pm|l,m\rangle\} = \hat{L}^\pm\hat{L}_z|l,m\rangle \pm \hbar\hat{L}^\pm|l,m\rangle$$
$$= (m\hbar \pm \hbar)\hat{L}^\pm|l,m\rangle$$
$$= (m\pm1)\hbar(\hat{L}^\pm|l,m\rangle)$$

So the effect of $\hat{L}^\pm$ is to raise or lower m by one unit.

Properly normalized: $\hat{L}^\pm|l,m\rangle = \{l(l+1) - m(m\pm1)\}^{1/2}\hbar^2|l,m\pm1\rangle$

It was experimentally determined in 1922 by Stern and Gerlach that there was more to angular momentum than just orbital motion:



beam should split into odd # of beams since $2\ell+1$ = odd if ℓ = integer.

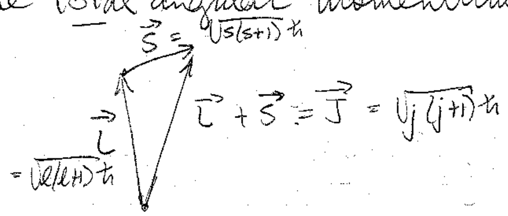
In fact, beam split in 2 $\Rightarrow \hbar/2$ for angular momentum.

So it was proposed that particles carried spin angular momentum in analogy with orbital angular momentum. We can define spin in an analogous way to orbital angular momentum

$$\hat{L}^2 |\ell, m_\ell\rangle = \ell(\ell+1)\hbar^2 |\ell, m_\ell\rangle$$
$$\hat{L}_z |\ell, m_\ell\rangle = m_\ell \hbar |\ell, m_\ell\rangle$$

$$\hat{S}^2 |s, m_s\rangle = s(s+1)\hbar^2 |s, m_s\rangle$$
$$\hat{S}_z |s, m_s\rangle = m_s \hbar |s, m_s\rangle .$$
$\hat{S}_\pm$ etc.

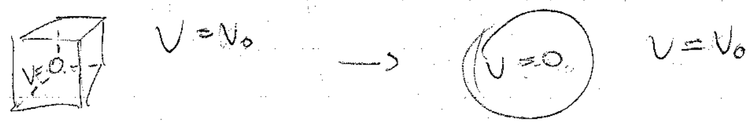
The total angular momentum is $\vec{J} = \vec{L} + \vec{S}$ added in a vector sense.



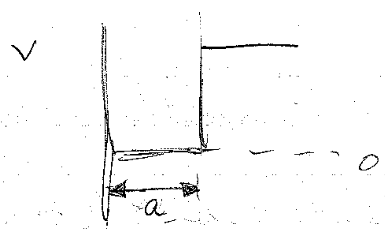
Lastly, we return to several problems with central forces.

1. Particle in a spherical finite well.

The problem is only slightly different than the "box" problem:



In spherical polar coordinates, this translates into



We solve

$$-\frac{\hbar^2}{2m}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \left\{-E + \left[V(r) + \frac{\hbar^2}{2mr^2}\ell(\ell+1)\right]\right\}R = 0$$

($V(r)$ ↳ 0 inside)

rewrite $R = \frac{u(r)}{r} \Rightarrow \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{du/r}{dr}\right) = \frac{1}{r^2}\frac{d}{dr}\left[r^2\,\frac{u'r-u}{r^2}\right] = \frac{1}{r^2}\left[u''r + u' - u'\right] = u''/r$

$$\Rightarrow -\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} = \left(E - \frac{\hbar^2}{2mr^2}\ell(\ell+1)\right)u \qquad \text{for } r < a$$

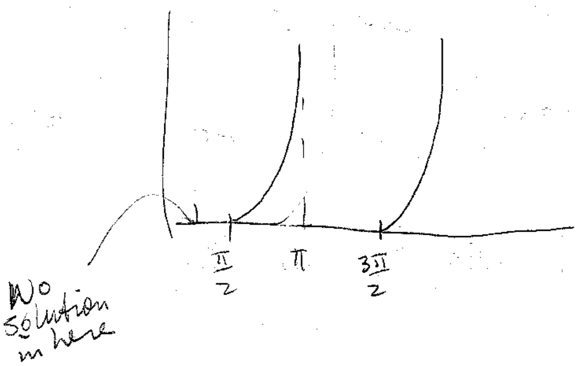
Say we start with $\ell = 0$ (s-waves only)

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} = Eu \quad \Rightarrow \quad \frac{d^2u}{dr^2} = -\frac{2mE}{\hbar^2}u.$$

This looks like the 1-D problem: $u = A\sin kr + B\cos kr$.
But now $\cos kr$ is not allowed since: $\psi = \frac{u}{r} = \frac{\cos kr}{r} \to \infty$ as $r \to 0$.

∴ the only solution left is $u = A \sin kr$.
Going through the usual procedure leads us to

$$\xi \cot \xi = -\eta$$



What's novel about the 3-D problem, then, is that

$\xi^2 + \eta^2 \geq \left(\frac{\pi}{2}\right)^2$ before there is a solution.

$\Rightarrow \; V_0 a^2 \geq \frac{\pi^2 \hbar^2}{8m}$ for $l=0$ sol'n.

Coulomb problem

Defining $\alpha^2 = -\frac{2mE}{\hbar^2}$ ($E < 0 \Rightarrow$ bound states).

$$\lambda = \frac{mZe^2}{\hbar^2}$$

$$\rho = 2\alpha r$$

$\Rightarrow$ r part of S.E. becomes $\frac{1}{\rho^2}\frac{d}{d\rho}\left(\rho^2 \frac{dS}{d\rho}\right) + \left(-\frac{1}{4} - \frac{l(l+1)}{\rho^2} + \frac{\lambda}{\rho}\right)S = 0$

Asymptotically, equation goes like $\frac{d^2S}{d\rho^2} = \frac{1}{4}S \Rightarrow S = e^{\pm\rho/2}$

Rewriting the equation like

$$S = e^{-\rho/2} F$$

$$\Rightarrow \frac{d^2}{d\rho^2}F + \left(\frac{2}{\rho} - 1\right)\frac{dF}{d\rho} + \left(\frac{\lambda}{\rho} - \frac{\ell(\ell+1)}{\rho^2} - \frac{1}{\rho}\right)F = 0$$

[confluent hypergeometric equation].

The solution is found by the power series expansion

$$F(\rho) = \rho^s L(\rho) \qquad L(\rho) = \sum_{i=0}^{\infty} a_i \rho^i$$

[generates associated Laguerre polynomials]

Substituting and demanding that the series for L be non-zero gives

$$s = \ell.$$

Then the rest of the series gives

$$a_{i+1} = \frac{(\lambda - \ell - 1 - i)}{[2(i+1)(\ell+1) + i(i+1)]} a_i$$

$$\Rightarrow \quad \lambda = \underbrace{i + \ell + 1}_{\substack{\ell = 0, 1, \ldots \\ i = 0, 1, \ldots}}$$

∴ replace sum with $n = 1, 2 \ldots$ (of course, we need i, ℓ for degeneracies etc)

The energies are then given by

$$\lambda = n \qquad (\text{Note, } \ell \leq n-1)$$

$$\Rightarrow E = -\frac{Z^2}{2n^2}\left(\frac{me^4}{\hbar^2}\right)$$

(If $a = \frac{\hbar^2}{Zme^2}$, then $E = -\frac{1}{2n^2}\frac{\hbar^2}{m}\left(\frac{Zme^2}{\hbar^2}\right)^2 = -\frac{1}{2n^2}\frac{\hbar^2}{m}\frac{1}{a^2}$.

Coulomb Potential

The radial equation which we have to solve is

$$\frac{1}{\rho^2}\frac{d}{d\rho}\left(\rho^2\frac{dS}{d\rho}\right)+\left(-\frac{1}{4}-\frac{\ell(\ell+1)}{\rho^2}+\frac{\lambda}{\rho}\right)S=0 \qquad (1)$$

where $\rho = 2\alpha r$ (and is ≥ 0).

$$\alpha^2 = -\frac{2mE}{\hbar^2} \qquad (E \leq 0 \text{ for bound states} \Rightarrow \alpha^2 \geq 0)$$

$$\lambda = \frac{mZe^2}{\hbar^2\alpha} \qquad \left(\text{where } V(r) = \frac{Ze^2}{r}\right)$$

The first term in (1) can be rewritten as:

$$\frac{1}{\rho^2}\frac{d}{d\rho}\left(\rho^2\frac{dS}{d\rho}\right) = \frac{1}{\rho^2}\left(2\rho\frac{dS}{d\rho}+\rho^2\frac{d^2S}{d\rho^2}\right) = \frac{2}{\rho}\frac{dS}{d\rho}+\frac{d^2S}{d\rho^2}.$$

Hence (1) becomes

$$\frac{d^2S}{d\rho^2}+\frac{2}{\rho}\frac{dS}{d\rho}-\frac{1}{4}S-\frac{\ell(\ell+1)}{\rho^2}S+\frac{\lambda}{\rho}S=0. \qquad (2)$$

To leading order in ρ, as $\rho \to \infty$ this becomes

$$\frac{d^2S}{d\rho^2}-\frac{1}{4}S=0.$$

This equation has the solution $S \propto e^{\pm\rho/2}$, where the $+\rho/2$ solution is discarded because it diverges.

We rewrite $S = e^{-\rho/2}F(\rho)$ as the complete solution, and find a differential equation for $F(\rho)$ by substitution:

Substituting into ②

$$\frac{d^2}{d\rho^2}\left(e^{-\rho/2}F\right) + \frac{2}{\rho}\frac{d}{d\rho}\left(e^{-\rho/2}F\right) + \left(-\frac{1}{4} - \frac{\ell(\ell+1)}{\rho^2} + \frac{\lambda}{\rho}\right)e^{-\rho/2}F = 0$$

$$\Rightarrow \frac{d}{d\rho}\left(-\frac{1}{2}e^{-\rho/2}F + e^{-\rho/2}F'\right) + \frac{2}{\rho}\left(-\frac{1}{2}e^{-\rho/2}F + e^{-\rho/2}F'\right)$$
$$+ \left(-\frac{1}{4} - \frac{\ell(\ell+1)}{\rho^2} + \frac{\lambda}{\rho}\right)e^{-\rho/2}F = 0$$

$$\Rightarrow \frac{1}{4}e^{-\rho/2}F - \frac{1}{2}e^{-\rho/2}F' - \frac{1}{2}e^{-\rho/2}F' + e^{-\rho/2}F'' - \frac{1}{\rho}e^{-\rho/2}F + \frac{2}{\rho}e^{-\rho/2}F'$$
$$+ \left(-\frac{1}{4} - \frac{\ell(\ell+1)}{\rho^2} + \frac{\lambda}{\rho}\right)e^{-\rho/2}F = 0$$

Removing the $e^{-\rho/2}$ and collecting terms, we find:

$$F'' + \left(\frac{2}{\rho} - 1\right)F' + \left(\frac{\lambda}{\rho} - \frac{\ell(\ell+1)}{\rho^2} - \frac{1}{\rho}\right)F = 0 \quad ③ \quad \left[\begin{array}{l} F'' = \frac{d^2}{d\rho^2}F \\ F' = \frac{d}{d\rho}F \end{array}\right]$$

We solve this by the usual power series method:

$$F(\rho) = \rho^s L \qquad , \; L = \sum_{i=0}^{n} a_i s^i$$

With this substitution,

$$F' = s\rho^{s-1}L + \rho^s L'$$

$$F'' = s(s-1)\rho^{s-2}L + s\rho^{s-1}L' + s\rho^{s-1}L' + \rho^s L''$$
$$= \rho^s L'' + 2s\rho^{s-1}L' + s(s-1)\rho^{s-2}L.$$

Let's substitute this into ③:

$$\rho^s L'' + 2s\rho^{s-1}L' + s(s-1)\rho^{s-2}L + \left(\frac{2}{\rho}-1\right)\left(s\rho^{s-1}L + \rho^s L'\right)$$
$$+\left(\frac{\lambda}{\rho} - \frac{\ell(\ell+1)}{\rho^2} - \frac{1}{\rho}\right)\rho^s L = 0$$

Multiplying by ρ^2 to get rid of possible singularities etc.

$$\rho^{s+2}L'' + \left(2s\rho^{s+1} + 2\rho^{s+1} - \rho^{s+2}\right)L'$$
$$+\underbrace{\left(s(s-1)\rho^s + \left(\frac{2}{\rho}-1\right)s\rho^{s+1} + \lambda\rho^{s+1} - \ell(\ell+1)\rho^s - \rho^{s+1}\right)}L = 0$$

$$(s^2 - s + 2s - \ell(\ell+1))\rho^s + (-s+\lambda-1)\rho^{s+1}$$
$$(s(s+1) - \ell(\ell+1))\rho^s + (\lambda - s - 1)\rho^{s+1}$$

Since L begins with a constant, a_0, then the lowest order term in ρ is

$$(s(s+1) - \ell(\ell+1))\rho^s L.$$

This must vanish for all ρ, hence $s = \ell$. We are left with

$$\rho^{\ell+2}L'' + (2\ell+2-\rho)\rho^{\ell+1}L' + (\lambda-\ell-1)\rho^{\ell+1}L = 0$$

or $\rho L'' + (2\ell+2-\rho)L' + (\lambda-\ell-1)L = 0$

Finally, substitute the series expansion for L and collect terms of order ρ^i:

$$\rho\, a_{i+1}(i+1)\, i\rho^{i-1} + 2(\ell+1)a_{i+1}(i+1)\rho^i - \rho a_i \cdot i\rho^{i-1}$$
$$+(\lambda-\ell-1)a_i\rho^i$$
$$= \left[a_{i+1}\cdot i \cdot i{+}1 + a_{i+1}\cdot 2\cdot(\ell+1)(i+1) - a_i \cdot i + (\lambda-\ell-1)a_i\right]\rho^i.$$

or $-\left\{\left[i(i+1)+2(i+1)(\ell+1)\right]a_{i+1}+(\lambda-\ell-1-i)a_i\right\}\rho^i$

This must vanish to satisfy the equation.

$$a_{i+1}=-\frac{(\lambda-\ell-1-i)}{[i(i+1)+2(i+1)(\ell+1)]}a_i$$

To make the series finite, $\lambda-\ell-1-i=0$

or $\lambda=\ell+i+1$ $\qquad i=0,1\ldots$ $\quad \ell=0,1\ldots$

call this $n=1,2,3\cdots$

$\ell\leq n-1$.

$$\lambda=\frac{mZe^2}{\hbar^2\alpha}=\frac{mZe^2}{\hbar^2}\left(-\frac{\hbar^2}{2mE}\right)^{1/2}$$

$$\Rightarrow\ \lambda^2=n^2=-\frac{m^2Z^2e^4}{\hbar^4}\frac{\hbar^2}{2mE}=-\frac{mZ^2e^4}{2E\hbar^2}\ \Rightarrow\ E=-\frac{mZ^2e^4}{2\hbar^2n^2}=-\frac{Z^2\hbar^2}{2mn^2}\frac{1}{a_0^2}$$

Some normalized radial wavefunctions are:

$R_{n\ell}$:

$$R_{10}=\left(\frac{Z}{a_0}\right)^{3/2}2e^{-\rho/2}$$

These ρ's are n-dependent: $a_0\equiv\frac{\hbar^2}{me^2}$, $\rho=2\alpha r$, $\alpha^2=-\frac{2mE}{\hbar^2}=+\frac{Z^2}{n^2a_0^2}$ $\Rightarrow \rho=2Zr/na_0$

These agree with Weider's solutions

$$R_{20}=\frac{(Z/a_0)^{3/2}}{2\sqrt{2}}(2-\rho)e^{-\rho/2}$$

$$R_{21}=\frac{(Z/a_0)^{3/2}}{2\sqrt{6}}\rho e^{-\rho/2}$$

$$R_{30}=\frac{(Z/a_0)^{3/2}}{9\sqrt{3}}(6-6\rho+\rho^2)e^{-\rho/2}$$

$$R_{31}=\frac{(Z/a_0)^{3/2}}{9\sqrt{6}}(4-\rho)\rho e^{-\rho/2}\qquad R_{32}=\frac{(Z/a_0)^{3/2}}{9\sqrt{30}}\rho^2e^{-\rho/2}.$$