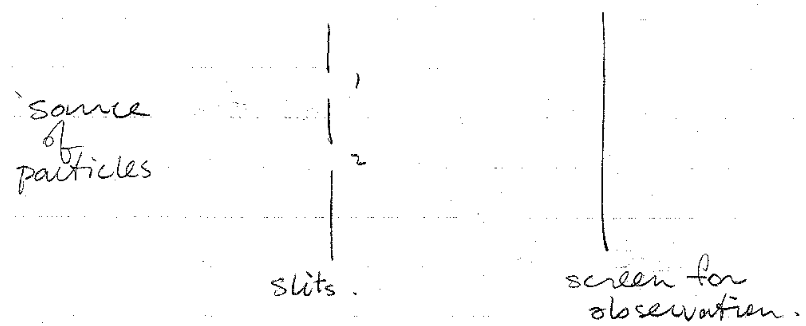
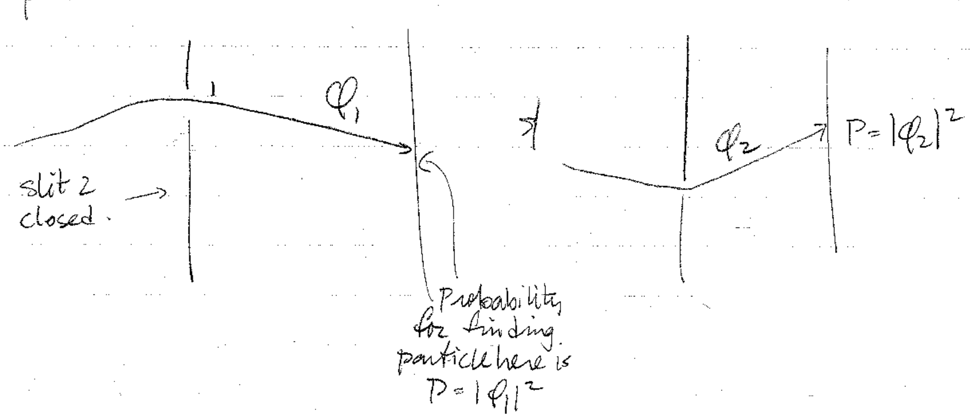
Path Integral Formalism (From Feynman and Hibbs).

The path integral formalism represents a different, but equivalent, approach to the description of motion at the quantum level than the state vector approach which we have used above. Let's begin, as Feynman does, with an example: interference in double slits:



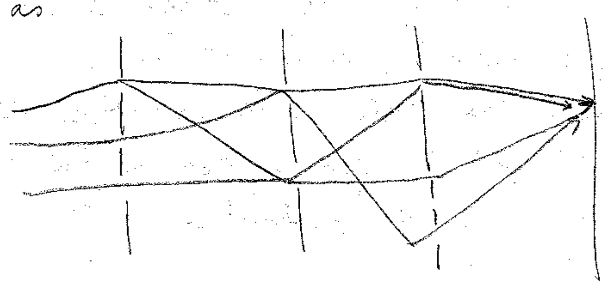
As before, we define a probability amplitude (like ψ) for the particle to pass through one of the slits and reach the screen at some position.



When the particle can pass through either slit, there are <u>interfering alternatives</u> and the probability of observation becomes

$$P \propto |\varphi_1 + \varphi_2|^2 .$$ (leaving out the $P\,dx$ etc).

One can consider a more complicated example such as

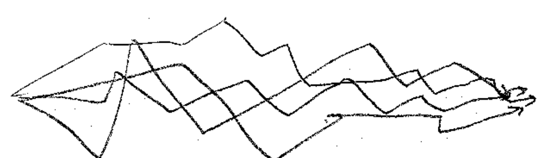


etc.

Now, the probability is just straightforwardly generalized to

$$P \propto \left| \sum_{\substack{\text{alternative} \\ \text{paths } i}} \varphi_i \right|^2 .$$

Let's pass now to the extreme limit of free particle motion. Then this problem is just like the slit one with an infinite # of slits cut into an infinite number of screens. The probability becomes



$$P \propto \left| \sum_{\substack{\text{all} \\ \text{paths}}} \varphi_i \right|^2 .$$

We need to find a way of constructing the individual amplitudes; to do this, we return to classical mechanics.

Classical Action

In our discussion of classical mechanics, we introduced the Lagrangian

$$L = T - V.$$

An associated quantity is the classical action

$$S = \int_{t_a}^{t_b} L(\dot{x}, x, t)\, dt.$$

One can show that the equations of motion correspond to the condition that the action be an extremum:

x_0, x_1 fixed, vary x in between:

$$S(x+\delta x) = \int_{t_0}^{t_1} L(\dot{x}+\delta\dot{x}, x+\delta x, t)\, dt$$

$$= \int_{t_0}^{t_1} \left[L(\dot{x}, x, t) + \delta\dot{x}\frac{\partial L}{\partial \dot{x}} + \delta x \frac{\partial L}{\partial x}\right] dt$$

$$= S(x) + \underbrace{\int_{t_0}^{t_1} \left(\delta\dot{x}\frac{\partial L}{\partial \dot{x}} + \delta x \frac{\partial L}{\partial x}\right) dt}$$

integrate by parts $\rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$

$+ S(\bar{x}+\delta x) - S(\bar{x}) = 0$

Let's give an example: $\uparrow$ classical path.

Free particle motion $\quad L = T - \underset{\rightarrow 0}{V} = \frac{m}{2}\dot{x}^2.$

$$S_{cl} = \int_{t_a}^{t_b} \frac{m}{2}\dot{x}^2\, dt = \frac{m}{2}\int_{t_a}^{t_b} \Big[\underbrace{\frac{(x_b - x_a)}{(t_b - t_a)}}_{\text{from eqn. of motion,}}\Big]^2 dt = \frac{m}{2}\frac{(x_b - x_a)^2}{(t_b - t_a)}$$

Quantum vs. Classical Paths:

Let's define a kernel $K(a,b)$ to go from a to b such that

$$K(b,a) = \sum_{\substack{\text{all paths} \\ \text{from } a \text{ to } b}} \varphi(x(t)).$$

We claim that the paths have a phase with respect to each other which goes like

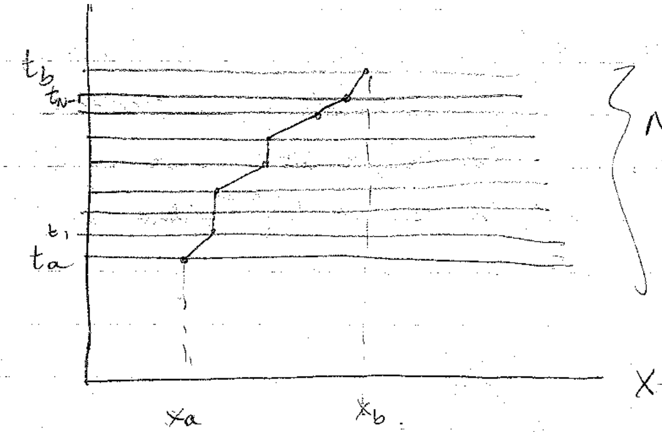
$$\varphi(x(t)) = \underbrace{\text{constant}}_{\text{to be discussed}} \, e^{\frac{i S(x(t))}{\hbar}}$$

First, we show that this regains us the classical picture where S is large compared to ħ:

Very near the classical path, S is an extremum and all paths have a similar phase. ∴ they add coherently. Away from the path, $S/\hbar$ changes rapidly and the paths interfere destructively. So we regain the classical limit for S large compared to ħ.

Sum → Integral

The tricky aspect of taking the sum over paths is finding the normalization constant such that the total probability is finite. Suppose that we break up the sum into:



At each time slice, we integrate over x:

$$K(b,a) \sim \int \cdots \int \phi[x(t)]\, dx_1\, dx_2 \cdots dx_{N-1}$$

(Note; no integral over x_a, x_b: they're fixed).

Defining a "normalizing factor" A we find

$$K(b,a) = \lim_{\epsilon \to 0} \frac{1}{A} \int \cdots \int e^{\frac{i}{\hbar} S[b,a]} \frac{dx_1}{A} \frac{dx_2}{A} \cdots \frac{dx_{N-1}}{A}$$

and

$$A = \left(\frac{2\pi i \hbar \epsilon}{m} \right)^{1/2}$$

$\leftarrow$ This is not a general definition. Note, N of them, not $N-1$

where

$$S[b,a] = \int_{t_a}^{t_b} L(\dot{x}, x, t)\, dt$$

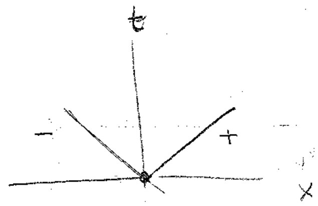
involves straight line segments, as in the figure.

Formally, we write

$$K(b,a) = \int_a^b e^{\frac{i}{\hbar} S[b,a]}\, \mathcal{D}x(t)$$

$\leftarrow$ path integral.

As an example of how to do the path evaluation, let us consider the relativistic example of a particle moving at the speed of light. We choose our units such that the particle moves along the light cone (1-D problem).



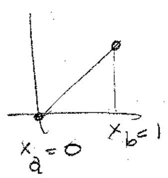
This has dimensions!

We can show that* $\varphi = (i\epsilon)^R$ for a given path, where $\epsilon = \frac{T}{N}$ $N = \#$ steps.

$R = \#$ of reversals along the path.

Let's consider several examples. Suppose $T = N\epsilon = 1$ (otherwise dimension problems)

Then for $x_b = 1$



only 1 path possible. $R = 0$.

$\Rightarrow N(i\epsilon)^R = 1$.

* Proof not given in Feynman up to page 35. Presumably, something like $L = \omega\hbar = \frac{2\pi\hbar}{T} =$ const.

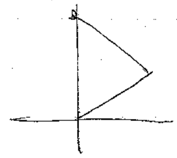
$e^{\frac{i}{\hbar}\int_0^\epsilon L\,dt} = e^{\frac{i}{\hbar}\frac{2\pi\hbar}{T}\epsilon} \cong 1 + \left(\frac{2\pi}{T}\right)i\epsilon$.

DON'T SCAN.

Now, let's put $x_b = 0$. Consider first, just paths on $x>0$ for first step.

Take $T=1$

$N(i\epsilon)^R$

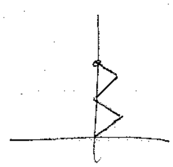
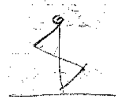


$x_a = x_b = 0$

If $\epsilon = \frac{T}{2} = \frac{1}{2}$, 1 reversal, 1 path. $i/2$

If $\epsilon = \frac{T}{4} = \frac{1}{4}$, 3 reversals, 1 path. $(i/4)^3$

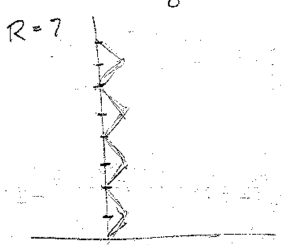
or 1 reversal, 1 path.  $(i/4)$

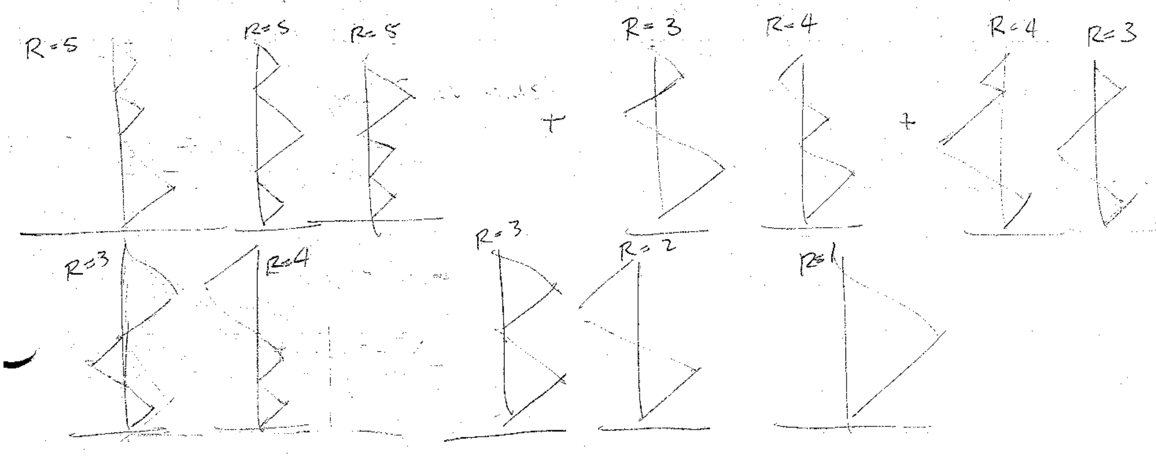
 +  1 path, 2 rev. $(i/4)^2$

If $\epsilon = \frac{T}{8}$

R	N	$N(i\epsilon)^R$
7	1	$1\,(\frac{i}{8})^7$
5	3	$3\,(i/8)^5$
4	3	$3\,(i/8)^4$
3	4	$4\,(i/8)^3$
2	1	$(i/8)^2$
1	1	$(i/8)$

In other words, # paths does not grow fast enough to acc't for ϵ decrease to power R. Also, phases begin to cancel.





Free particle motion (FH, pg. 42) in 1-D.

The Lagrangian has the simple form

$$L = m\frac{\dot{x}^2}{2}$$

The kernel for the free particle, including the normalization constant, is of the form

$$K(b,a) = \lim_{\epsilon \to 0} \int \cdots \int \exp\left(\frac{iS_{cl}}{\hbar}\right) dx_1 \cdots dx_{N-1} \left(\frac{2\pi i \hbar \epsilon}{m}\right)^{-N/2}$$

from before,

$$\frac{m}{2\,\Delta t}(x_i - x_{i-1})^2 \qquad \Delta t = \epsilon$$

Let's start with the x_1 integral. Define

$$I_1 = \left(\frac{2\pi i\hbar\epsilon}{m}\right)^{-1/2}\left(\frac{2\pi i\hbar\epsilon}{m}\right)^{-1/2}\int_{-\infty}^{\infty} \exp\left\{\frac{m}{2i\hbar\epsilon}\left[(x_2-x_1)^2+(x_1-x_0)^2\right]\right\} dx_1$$

We have N of these in total, but $N-1$ integrals. For reasons which become clear in a moment, take two together.

$$x_2^2 - 2x_1x_2 + x_1^2 + x_1^2 - 2x_1x_0 + x_0^2$$
$$= 2x_1^2 - 2x_1(x_2+x_0) + (x_0^2+x_2^2)$$
$$= 2\left[x_1^2 - 2x_1\left(\tfrac{1}{2}(x_2+x_0)\right) + \tfrac{1}{4}(x_2+x_0)^2 - \tfrac{1}{4}(x_2+x_0)^2\right] + x_0^2 + x_2^2$$
$$= 2\left[x_1 - \tfrac{1}{2}(x_2+x_0)\right]^2 - \frac{(x_2+x_0)^2}{2} + x_0^2 + x_2^2$$

$$\frac{-x_2^2 - 2x_0x_2 - x_0^2 + 2x_0^2 + 2x_2^2}{2}$$
$$= \frac{x_0^2 - 2x_0x_2 + x_2^2}{2} = \frac{(x_2-x_0)^2}{2}.$$

$$I_1 = \left(\frac{2\pi i\hbar\epsilon}{m}\right)^{-2/2} \exp\left[\frac{-m}{2i\hbar\epsilon}\frac{(x_2-x_0)^2}{2}\right]\int \exp\left\{\frac{-m}{2i\hbar\epsilon}2\left[x_1-\frac{x_0+x_2}{2}\right]^2\right\}dx$$

$$= \left(\frac{2\pi i\hbar\epsilon}{m}\right)^{-2/2} \exp\left[\frac{-m}{2i\hbar\epsilon}\frac{(x_2-x_0)^2}{2}\right]\int_{-\infty}^{\infty} \exp\left(\frac{-m}{i\hbar\epsilon}y^2\right)dy$$

$$= \left(\frac{2\pi i\hbar\epsilon}{m}\right)^{-2/2} \exp\left[\frac{-m}{2i\hbar(2\epsilon)}(x_2-x_0)^2\right]\left(\frac{-m}{i\hbar\epsilon}\right)^{-1/2}\underbrace{\int_{-\infty/2}^{\infty/2} \exp(-z^2)dz}_{\sqrt{\pi}}.$$

$$= \left(\frac{2\pi i\hbar\epsilon}{m}\right)^{-2/2}\left(\frac{m}{\pi i\hbar\epsilon}\right)^{-1/2} \exp\left(\frac{-m}{2i\hbar(2\epsilon)}(x_2-x_0)^2\right)$$

$$= \left(\frac{2\pi i\hbar(2\epsilon)}{m}\right)^{-1/2} \exp\left(\frac{-m}{2i\hbar(2\epsilon)}(x_2-x_0)^2\right).$$

Bringing in the next factor so that x_2 can be integrated out:

$$I_2 = \left(\frac{2\pi i\hbar\epsilon}{m}\right)^{-2/2}\frac{1}{\sqrt{2}}\int \exp\left\{\frac{-m}{2i\hbar\epsilon}\left[(x_3-x_2)^2+\frac{(x_2-x_0)^2}{2}\right]\right\}dx$$

$$\frac{1}{2}\left[2(x_3^2-2x_2x_3+x_2^2)+x_2^2-2x_2x_0+x_0^2\right]$$

$$= \frac{1}{2}\left[2x_3^2-4x_2x_3+2x_2^2+x_2^2-2x_2x_0+x_0^2\right]$$

$$= \frac{1}{2}\left[3x_2^2-2x_2(x_0+2x_3)+x_0^2+2x_3^2\right]$$

$$= \frac{1}{2}\left[3\left(x_2^2-2x_2\left(\frac{x_0+2x_3}{3}\right)+\left(\frac{x_0+2x_3}{3}\right)^2-\left(\frac{x_0+2x_3}{3}\right)^2\right]+x_0^2+2x_3^2\right]$$

$$= \frac{1}{2}\left\{3\left[x_2-\frac{x_0+2x_3}{3}\right]^2 - \frac{(x_0+2x_3)^2}{3}+x_0^2+2x_3^2\right\}$$

$$\frac{1}{3}\left[-x_0^2-4x_0x_3-4x_3^2+3x_0^2+6x_3^2\right] = \frac{1}{3}\left[2x_0^2-4x_0x_3+2x_3^2\right]$$

$$= \frac{2}{3}\left[x_0-x_3\right]^2 \leftarrow \text{Then times } \underline{\underline{1/2}}\{\dots\}$$

$$\Rightarrow I_2 = \left(\frac{2\pi i\hbar\epsilon}{m}\right)^{-2/2} \frac{1}{\sqrt{2}} \exp\left[\frac{-m}{2i\hbar\epsilon}\cdot\frac{1}{2}\cdot\frac{2}{3}(x_3-x_0)^2\right] \times$$

$$\times \int \exp\left[\frac{-m}{2i\hbar\epsilon}\left(\frac{3}{2}\right)\left(x_2 - \frac{x_0+2x_3}{3}\right)^2\right] dx_2$$

$$= \left(\frac{2\pi i\hbar\epsilon}{m}\right)^{-2/2} \frac{1}{\sqrt{2}} \exp\left(\frac{-m}{2i\hbar(3\epsilon)}(x_3-x_0)^2\right)\left(\exp\frac{m}{2i\hbar\epsilon}\left(\frac{3}{2}\right) \cdots\right.$$

This was 2ϵ before.

These give rise to $\sqrt{2}\cdot\sqrt{\frac{3}{2}}$

i.e. $2\epsilon \to 3\epsilon$ and $x_2 \to x_3$ in going from $I_1 \to I_2$.

So, the general expression becomes

$$I_{N-1} = \left(\frac{2\pi i\hbar(N\epsilon)}{m}\right)^{-1/2} \exp\left(\frac{-m}{2i\hbar(N\epsilon)}(x_n-x_0)^2\right).$$

But $N\epsilon = t_b - t_a$.

$$\Rightarrow K(b,a) = \left[\frac{2\pi i\hbar(t_b-t_a)}{m}\right]^{-1/2} \exp\left(\frac{-m(x_b-x_a)^2}{2i\hbar(t_b-t_a)}\right)$$

(This disagrees with 3-3 in F&H, who have an "i" in the numerator of the exponential).

Now, if $P(b,a)dx$ is the probability of the particle arriving at b from a, then

$$P(b,a)\,dx = |K(b,a)|^2\,dx = \left[\frac{m}{2\pi\hbar(t_b-t_a)}\right] dx$$

If we define $t_a = x_a = 0$, then of course

$$K(x,t;0,0) = \left(\frac{2\pi i \hbar t}{m}\right)^{-1/2} \exp\left(\frac{imx^2}{2\hbar t}\right)$$