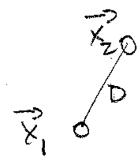
Kinematics and Dynamics (Ch. 3)

Transformations of states and observables

The idea that the laws of nature are invariant under certain types of symmetry operations has been around for some time. Take a simple measurement like distance



$$D^2 = (\vec{x}_1 - \vec{x}_2)^2$$

Even though we may make a coordinate transformation on $\vec{x}_1$ & $\vec{x}_2 \longrightarrow \vec{x}_1{}', \vec{x}_2{}'$, we still expect D^2 to be the same

$$(\vec{x}_1 - \vec{x}_2)^2 = (\vec{x}_1{}' - \vec{x}_2{}')^2 .$$

Let's rework this in terms of our vector notation:

$|\varphi_n\rangle \longrightarrow |\varphi_n'\rangle$ (new coordinates)

$\hat{A} \longrightarrow \hat{A}'$ (operator expressed in new coordinates).

1. Invariance: $\hat{A}'|\varphi_n'\rangle = a_n |\varphi_n'\rangle$ if $\hat{A}|\varphi_n\rangle = a_n|\varphi_n\rangle$

These are the same.

2. If we have $|\psi\rangle = \sum_n c_n |\varphi_n\rangle$ where $|\varphi_n\rangle$ are eigenvectors of $\hat{A}$ then

$|\psi'\rangle = \sum_n c_n' |\varphi_n'\rangle$ where $|\varphi_n'\rangle$ are eigenvectors of $\hat{A}'$.

Now, if events have the same probability in either frame of reference, then $|c_n|^2 = |c_n'|^2$ [eigenvalues don't change].

$\Rightarrow |\langle\varphi_n|\psi\rangle|^2 = |\langle\varphi_n'|\psi'\rangle|^2$ by definitions of c_n

Theorem

Any mapping of the vector space onto itself which preserves the value of $|\langle\varphi|\psi\rangle|$ (not $\langle\varphi|\psi\rangle$, just its magnitude) may be implemented by a unitary or antiunitary operator

$$|\psi\rangle \to |\psi'\rangle = \hat{U}|\psi\rangle$$
$$|\varphi\rangle \to |\varphi'\rangle = \hat{U}|\varphi\rangle$$

e.g., for the unitary case $\hat{U}^+\hat{U} = \hat{U}\hat{U}^+ = \hat{I}$

$$\Rightarrow \langle\varphi'|\psi'\rangle = (\langle\varphi|\hat{U}^+)(\hat{U}|\psi\rangle) = \langle\varphi|\psi\rangle$$

[The antiunitary operator takes $\langle\varphi'|\psi'\rangle \to \langle\varphi|\psi\rangle^*$

Only unitary operators describe continuous transformations (although antiunitary operators are used for certain discrete symmetries). For proof, assume the converse:

$$\hat{U}(2\ell) = \hat{U}(\ell)\hat{U}(\ell)$$

If each of these is antiunitary, then each produces a complex conjugate. But the product of the c.c. vanishes, i.e. $\hat{U}(2\ell)$ does not involve a c.c. (it's unitary) But $\hat{U}(\ell)$ should not be unitary for some ℓ (i.e. 2ℓ) and antiunitary for others (ℓ).
∴ it must be unitary $\forall \ell$.

Now, let's find the relationship between $\hat{A}'$ and $\hat{A}$:
We demand that $\hat{A}'|\varphi_n'\rangle = a_n|\varphi_n'\rangle$ if $\hat{A}|\varphi_n\rangle = a_n|\varphi_n\rangle$

$$\hat{A}'\hat{U}|\varphi_n\rangle = a_n\hat{U}|\varphi_n\rangle$$

operate with $\hat{U}^{-1}$ $\Rightarrow \hat{U}^{-1}\hat{A}'\hat{U}|\varphi_n\rangle = a_n\hat{U}^{-1}\hat{U}|\varphi_n\rangle = a_n|\varphi_n\rangle = \hat{A}|\varphi_n\rangle$.

$$\therefore \quad \hat{U}^{-1}\hat{A}'\hat{U} = \hat{A}$$

or $\boxed{\hat{A}' = \hat{U}\hat{A}\hat{U}^{-1}}$ ** (note change in order).

Next, let's see how we can represent $\hat{U}$ so that it satisfies the previous constraint. Suppose $\hat{U}$ is written as a function of some parameter s, which is continuous: $\hat{U} = \hat{U}(s)$ such that

$$\hat{U}(s) \to \hat{I} \text{ as } s \to 0$$
$$\hat{U}(s_1 + s_2) = \hat{U}(s_1)\,\hat{U}(s_2).$$

For small s, we use a Taylor-like expansion for the operator:

$$\hat{U}(s) = \hat{I} + \left.\frac{d\hat{U}}{ds}\right|_{s=0} \cdot s + \text{order } s^2.$$

But unitarity requires

$$\hat{I} = UU^\dagger = \left(\hat{I} + \left.\frac{d\hat{U}}{ds}\right|_{s=0} \cdot s + \ldots\right)\left(\hat{I} + \left.\frac{d\hat{U}^\dagger}{ds}\right|_{s=0} \cdot s + \ldots\right)$$
$$= \hat{I} + \left(\frac{d\hat{U}}{ds} + \frac{d\hat{U}^\dagger}{ds}\right) \cdot s + \text{order } s^2.$$

To make the identity hold, we require all terms vanish for arbitrary s:

$$\frac{d\hat{U}}{ds} + \left.\frac{d\hat{U}^\dagger}{ds}\right|_{s=0} = 0$$

We can satisfy this with

$$\left.\frac{d\hat{U}}{ds}\right|_{s=0} = i\hat{k}$$

where $\hat{k} = \hat{k}^\dagger$ is referred to as the <u>generator</u> of the operators.

To find the general solution for U, set up a differential equation:

$$\hat{U}(s_1 + s_2) = \hat{U}(s_1)\,\hat{U}(s_2).$$

$$\Rightarrow \left.\frac{\partial}{\partial s_2} U(s_1 + s_2)\right|_{s_2 \to 0} = \left[\hat{U}(s_1)\frac{d}{ds_2}\hat{U}(s_2)\right]_{s_2 \to 0} = \hat{U}(s_1)\, i\hat{k}$$

But this is valid for any s_1, so taking the $s_2 = 0$ limit

$$\left.\frac{dU(s)}{ds}\right|_{s=s_1} = U(s_1)\, ik \quad \Rightarrow \quad \hat{U} = e^{i\hat{k}s}$$

[This uses the boundary condition $\hat{U}(0) = \hat{I}$]

Galilei group

Let's look at how these transformations actually appear, so we can find the forms of the generators. The group of transformations underlying Newtonian mechanics is the Galilei group:

$$\vec{x} \rightarrow \vec{x}' = \underset{\substack{\uparrow \\ \text{rotation}}}{R\vec{x}} + \underset{\substack{\uparrow \\ \text{translation} \\ \text{of origin}}}{\vec{a}} + \vec{v}t$$

↖ moving reference frame

$$t \rightarrow t' = t + \underset{\substack{\uparrow \\ \text{shift of time origin}}}{s}$$

There are $\underline{\underline{10}}$ parameters in this group: $R \rightarrow 3\times3$ matrix $\Rightarrow$ 3 Euler angles
$\vec{a} \rightarrow$ 3 component vector
$\vec{v} \rightarrow$ 3 component vector
$\vec{s} \rightarrow$ 1 scalar.

For a general transformation with parameter set τ, we write

$$U(\tau) = \prod_{i=1}^{10} e^{is_\mu \hat{k}_\mu}$$

Consider the product of 2 k's, say $U(s_1, s_2) = e^{i\hat{k}_1 s_1} e^{i\hat{k}_2 s_2}$
Let's take $s_1 = s_2 = \epsilon$, then form $U(-\epsilon)U(\epsilon) \neq UU^{-1}$

$$\hat{U}(\epsilon)\hat{U}(-\epsilon) = e^{i\epsilon k_1} e^{i\epsilon k_2} e^{-i\epsilon k_1} e^{-i\epsilon k_2} \simeq (\hat{I} + i\epsilon\hat{k}_1 - \frac{\epsilon^2 \hat{k}_1^2}{2} \ldots)\times$$
$$(\hat{I} + i\epsilon\hat{k}_2 - \frac{\epsilon^2\hat{k}_2^2}{2} \ldots)\times$$
$$(\hat{I} - i\epsilon\hat{k}_1 - \frac{\epsilon^2\hat{k}_1^2}{2} \ldots)\times$$
$$(\hat{I} - i\epsilon\hat{k}_2 - \frac{\epsilon^2\hat{k}_2^2}{2} \ldots)$$
$$= \hat{I} + i\epsilon(\hat{k}_1 + \hat{k}_2 - \hat{k}_1 - \hat{k}_2) + \epsilon^2(-\hat{k}_1\hat{k}_2 + \hat{k}_1^2 + \hat{k}_1\hat{k}_2 + \hat{k}_2\hat{k}_1 + \hat{k}_2^2 - \hat{k}_1\hat{k}_2 - 2\frac{\hat{k}_1^2}{2} - 2\frac{\hat{k}_2^2}{2}) + O(\epsilon^3)$$
$$= \hat{I} + \epsilon^2(\hat{k}_2\hat{k}_1 - \hat{k}_1\hat{k}_2) + O(\epsilon^3)\ldots \quad (1)$$

The operation of the displacement and its inverse should be ~~bring back $|\psi\rangle$ to itself within a phase~~ equivalent to within a phase $e^{i\omega}$ of some set of parameters (s_1, s_2) with

$$e^{i\omega}\,\hat{U} = \hat{I} + i\sum_{1,2} s_\mu \hat{K}_\mu + i\omega\hat{I} \qquad (2)$$

[That is $x_\mu \xrightarrow{\epsilon} x_\mu' \xrightarrow{-\epsilon} x_\mu$ should be equivalent to $x \circlearrowleft s$ to within a phase $e^{i\omega}$].

This means that the commutator in Eq. 1 should have the form (generalizing $1,2 \to \mu,\nu$)

$$[\hat{K}_\mu, \hat{K}_\nu] = i\sum_\lambda C^\lambda_{\mu\nu}\hat{K}_\lambda + i\underbrace{b_{\mu\nu}}_{\text{phase}}\hat{I}.$$

The coefficients $C^\lambda_{\mu\nu}$ and $b_{\mu\nu}$ are constants to be determined (in a few minutes). Before we do that, we break up the 10 generators $\hat{K}$ ~~and relabel~~ into groups and relabel them:

Rotations
$\vec{x} \to R_\alpha(\theta_\alpha)\vec{x}$ (angle: θ_α) $\quad \alpha = 1,2,3 \quad e^{-i\theta_\alpha J_\alpha}$ $\quad$ θ - parameter, J - generator

Displacement
$\vec{x} \to \vec{x} + \vec{a}_\alpha \quad \alpha = 1,2,3 \quad e^{-ia_\alpha P_\alpha}$

Velocity
$x_\alpha \to x_\alpha + v_\alpha t \quad \alpha = 1,2,3 \quad e^{iv_\alpha G_\alpha}$

Time displacement
$t \to t + s \quad e^{isH}$ $\quad$ (Note sign convention)

Evaluation of the commutators

To find the commutators of $\hat{J}_\alpha$, $\hat{P}_\alpha$, $\hat{G}_\alpha$ and $\hat{H}$, we have to perform some explicit transformations. Some of these are straightforward and we do them now:

Pure spatial displacements commute with each other; it doesn't matter what order the shift $x \xrightarrow{a_1} x' \xrightarrow{a_2} x''$ is done in. Hence:

$$[\hat{P}_\alpha, \hat{P}_\beta] = 0 + \xi \hat{I}$$

where ξ is an unknown constant to be determined later.

Similarly, spatial (x) and temporal (t) translations commute

$$[\hat{P}_\alpha, \hat{H}] = 0 + \xi \hat{I}.$$

We also have velocities commuting among themselves and velocities commuting with spatial displacements (up to factors $\hat{I}$):

$$[\hat{P}_\alpha, \hat{G}_\beta] = 0 + \xi \hat{I}$$

$$[\hat{G}_\alpha, \hat{G}_\beta] = 0 + \xi \hat{I}.$$

Moving on to rotations, rotations commute with time

$$[\hat{J}_\alpha, \hat{H}] = 0 + \xi \hat{I}$$

and rotations commute with a displacement or velocity shift <u>along</u> the axis of rotation:

$$[\hat{J}_\alpha, \hat{P}_\alpha] = 0 + \xi \hat{I}$$

$$[\hat{J}_\alpha, \hat{G}_\alpha] = 0 + \xi \hat{I}.$$

Let's stop and see how far we've got:

$\hat{P}$	0			
$\hat{G}$	0	0		
$\hat{J}$	$\alpha\alpha=0$	$\alpha\alpha=0$		
$\hat{H}$	0		0	0
	$\hat{P}$	$\hat{G}$	$\hat{J}$	$\hat{H}$

$+ \xi \hat{I}$

Let's now look at $[\hat{G}, \hat{H}]$. Suppose we transform $v_x \to v_x - \epsilon$, then $t \to t - \sigma$, followed by their inverse: ϵ, σ small.

$$e^{i\sigma\hat{H}} e^{i\epsilon\hat{G}_1} e^{-i\sigma\hat{H}} e^{-i\epsilon\hat{G}_1} = \hat{I} + \sigma\epsilon [\hat{G}_1, \hat{H}] \cdots$$

The effect of the transformations is

$$\begin{aligned}(x_1, x_2, x_3; t) &\to (x_1 - \epsilon t, x_2, x_3; t) \\ &\to (x_1 - \epsilon t, x_2, x_3; t - \sigma) \\ &\to (x_1 - \epsilon t + \epsilon(t - \sigma), x_2, x_3; t - \sigma) \\ &\to (x_1 - \sigma\epsilon, x_2, x_3; t).\end{aligned}$$

This is just a spatial displacement $-\sigma\epsilon$ along x-axis, which is $\underline{\underline{e^{+i\hat{P}_1\sigma\epsilon}}} = \hat{I} + i\sigma\epsilon\hat{P}_1$.

Note sign convention

$\therefore$ $[\hat{G}_1, \hat{H}] = i\hat{P}_1 + \xi\hat{I}$.

[similarly with $\hat{G}_2$, $\hat{G}_3$].

Let's move on to rotations. A rotation consists of a coordinate transformation like

$$x_i \to \sum_{j=1}^{3} R_{ij} x_j$$ where R_{ij} is a rotation matrix.

The rotation matrices have the form:

$$R_1(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \quad R_2(\theta) = \begin{bmatrix} \cos & 0 & +\sin\theta \\ 0 & 1 & 0 \\ -\sin & 0 & \cos\theta \end{bmatrix} \quad R_3(\theta) = \begin{bmatrix} \cos & -\sin & 0 \\ \sin & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(note sign convention)

Now, these can be recast using the series expressions

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \cdots \qquad \cos\theta = 1 - \frac{\theta^2}{2!} \cdots$$

$$R_\alpha(\theta) = e^{-i\theta M_\alpha}$$

Where

$$M_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix} \qquad M_2 = \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{bmatrix} \qquad M_3 = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

For example, for $\theta = \epsilon$ (small):

$$R_1(\epsilon) = e^{-i\epsilon M_1} = 1 + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i(-i) \\ 0 & -i(i) & 0 \end{bmatrix}\epsilon + \cdots = 1 + \epsilon \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \cdots$$

Similarly,

$$R_2(\epsilon) = e^{-i\epsilon M_2} = 1 + \epsilon \begin{bmatrix} 0 & 0 & -i(i) \\ 0 & 0 & 0 \\ -i(-i) & 0 & 0 \end{bmatrix} = 1 + \epsilon \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

To find commutators, we evaluate $R_2(-\epsilon)\, R_1(-\epsilon)\, R_2(\epsilon)\, R_1(\epsilon)$

$$= \left(1 + i\epsilon M_2 - \frac{\epsilon^2}{2} M_2^2\right)\left(1 + i\epsilon M_1 - \frac{\epsilon^2}{2} M_1^2\right)\left(1 - i\epsilon M_2 - \frac{\epsilon^2}{2} M_2^2 \cdots\right)\left(1 - i\epsilon M_1 - \frac{\epsilon^2}{2} M_1^2 \cdots\right)$$

$$= \left(1 + i\epsilon M_2 + i\epsilon M_1 - \epsilon^2 M_2 M_1 - \frac{\epsilon^2}{2} M_2^2 - \frac{\epsilon^2}{2} M_1^2\right) \times$$

$$\times \left(1 - i\epsilon M_2 - i\epsilon M_1 - \epsilon^2 M_2 M_1 - \frac{\epsilon^2}{2} M_2^2 - \frac{\epsilon^2}{2} M_1^2\right)$$

$$= 1 + \underbrace{i\epsilon \times 0}_{\text{sum vanishes}} + \epsilon^2\left(1 - \tfrac{1}{2} - \tfrac{1}{2}\right)M_1^2 + \epsilon^2\left(1 - \tfrac{1}{2} - \tfrac{1}{2}\right)M_2^2 - 2\epsilon^2 M_2 M_1 + \epsilon^2 M_1 M_2 + \epsilon^2 M_2 M_1$$

$$= \mathbb{1} + \epsilon^2 (M_1 M_2 - M_2 M_1) \cdots$$

(Done on page 3-9)

$$= \mathbb{1} + \epsilon^2 \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix} \right\}$$

$$= \mathbb{1} + \epsilon^2 \left\{ \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\} = \mathbb{1} + \epsilon^2 \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \mathbb{1} + \epsilon^2 i \mathcal{M}_3 .$$

$$= R_3(-\epsilon^2)$$

$$\Rightarrow \quad e^{i\epsilon \hat{J}_2} e^{i\epsilon \hat{J}_1} e^{-i\epsilon \hat{J}_2} e^{-i\epsilon \hat{J}_1} = e^{i\omega} e^{i\epsilon^2 \hat{J}_3}$$

$$\Rightarrow [\hat{J}_1, \hat{J}_2] = i\hat{J}_3 + \xi \hat{I} .$$

One can generalize this to $[\hat{J}_\alpha, \hat{J}_\beta] = i\epsilon_{\alpha\beta\gamma} \hat{J}_\gamma + \xi \hat{I}$

where $\epsilon_{\alpha\alpha\beta} = 0 \quad \epsilon_{123} = +1$
odd permutations $= -1$.

Combinations of rotations and velocity or position changes become

$$[\hat{J}_\alpha, \hat{G}_\beta] = i\epsilon_{\alpha\beta\gamma} \hat{G}_\gamma$$

$$[\hat{J}_\alpha, \hat{P}_\beta] = i\epsilon_{\alpha\beta\gamma} \hat{P}_\gamma .$$

Multiples of the Identity : ξ

Now, we turn to the phase question, what is the phase factor $e^{i\omega}$ or, equivalently, what is ξ?

There are several techiques for determining whether a multiple of the identity is present.

a) For some commutators, one can show that the multiple vanishes explicitly. For example, an operator <u>must</u> commute with itself:

$$[\hat{P}_\alpha , \hat{P}_\alpha] = \xi \hat{I}$$
$$\hookrightarrow = \hat{P}_\alpha \hat{P}_\alpha - \hat{P}_\alpha \hat{P}_\alpha = 0 \Rightarrow \xi = 0.$$

In other cases, one can use the Jacobi identity to show $\xi = 0$:

$$[[\hat{K}_\mu , \hat{K}_\nu], \hat{K}_\lambda] + [[\hat{K}_\nu , \hat{K}_\lambda], \hat{K}_\mu] + [[\hat{K}_\lambda , \hat{K}_\mu], \hat{K}_\nu] = 0$$

For example:

$$[[\hat{G}_1 , \hat{H}], \hat{P}_3] + [[\hat{H} , \hat{P}_3], \hat{G}_1] + [[\hat{P}_3 , \hat{G}_1], \hat{H}] = 0$$

$$\Rightarrow [(i\hat{P}_1 + \xi_1 \hat{I}), \hat{P}_3] + \underbrace{[\xi_2 \hat{I} , \hat{G}_1]}_{=0} + \underbrace{[\xi_3 \hat{I} , \hat{H}]}_{=0} = 0$$

$$\Rightarrow i[\hat{P}_1 , \hat{P}_3] + 0 + 0 + 0 = 0 \quad \Rightarrow [\hat{P}_1 , \hat{P}_3] = 0.$$

One can similarly show that

$$[\hat{P}_\alpha , \hat{H}] = 0$$
$$[\hat{G}_\alpha , \hat{G}_\beta] = 0$$
$$[\hat{J}_\alpha , \hat{H}] = 0.$$

b) For some commutators, z can be absorbed in an arbitrary phase:

Take, for example, the rotation generators J;

$$[\hat{J}_\alpha, \hat{J}_\beta] = i\epsilon_{\alpha\beta\gamma} J_\gamma + z\hat{I}$$

This must be of the form $i\epsilon_{\alpha\beta\gamma} b_\gamma \hat{I}$

Now, if we make the transformation

$$\hat{J}_\alpha + b_\alpha \hat{I} \rightarrow \hat{J}_\alpha' ,$$

Then the commutation relation for J' reads

$$[\hat{J}_\alpha', \hat{J}_\beta'] = i\epsilon_{\alpha\beta\gamma} J_\gamma'$$

What is the significance of the term b_α? It just introduces a new phase into the state $|\psi\rangle$ after rotation:

$$e^{-iJ_\alpha'\theta} |\psi\rangle = e^{-ib_\alpha\theta} e^{-i\hat{J}_\alpha\theta} |\psi\rangle$$

This extra phase is not significant. If we operate $J'(-\theta)J'(\theta)$ it will disappear.

There are similar phases which can be removed in the commutators $[J, G]$ and $[J, P]$.

c) The last commutator which we are left with is $[G, P]$ (velocity - position). Using the Jacobi identity, we can show that $[\hat{G}_\alpha, \hat{P}_\beta] = 0 \quad \alpha \neq \beta$. What about $\alpha = \beta$:

Try
$$[\underbrace{[\hat{J}_1, \hat{G}_2]}_{i\hat{G}_3}, \hat{P}_3] + [\underbrace{[\hat{G}_2, \hat{P}_3]}_{0}, \hat{J}_1] + [\underbrace{[\hat{P}_3, \hat{J}_1]}_{i\hat{P}_2}, \hat{G}_2] = 0$$

$$\Rightarrow \quad i[\hat{G}_3, \hat{P}_3] + i[\hat{P}_2, \hat{G}_2] = 0$$

$$\text{or} \quad [\hat{P}_2, \hat{G}_2] = [\hat{P}_3, \hat{G}_3].$$

In fact, we have no way of eliminating the extra ξI in $[P_\alpha, G_\alpha]$ if we use only commutation relations. Hence, we write

$$[\hat{G}_\alpha, \hat{P}_\alpha] = i\delta_{\alpha\beta} M \hat{I}$$

where M is an unknown constant.

So, to summarize, we have:

$$[\hat{P}_\alpha, \hat{P}_\beta] = 0 \qquad [\hat{P}_\alpha, \hat{H}] = 0$$
$$[\hat{G}_\alpha, \hat{G}_\beta] = 0 \qquad [\hat{G}_\alpha, \hat{H}] = iP_\alpha$$
$$[\hat{J}_\alpha, \hat{J}_\beta] = i\epsilon_{\alpha\beta\gamma}\hat{J}_\gamma \qquad [\hat{J}_\alpha, \hat{H}] = 0$$

$$[\hat{J}_\alpha, \hat{P}_\beta] = i\epsilon_{\alpha\beta\gamma}\hat{P}_\gamma$$
$$[\hat{J}_\alpha, \hat{G}_\beta] = i\epsilon_{\alpha\beta\gamma}\hat{G}_\gamma$$
$$[\hat{G}_\alpha, \hat{P}_\beta] = i\delta_{\alpha\beta} M \hat{I}.$$

Note that up to this point there has been no mention of uncertainty principles, representations, etc. Just commutators.

Identifying operators with Dynamical Observables

We start with the free particle situation – no spin; no external interactions, no other particles. Let's define a position operator $\hat{\vec{Q}}$ such that for position eigenkets $|\vec{x}\rangle$,

$$\hat{Q}_\alpha |\vec{x}\rangle = x_\alpha |\vec{x}\rangle \qquad \alpha = 1,2,3$$

$\vec{x}$ continuous.

Now, we want to introduce a velocity operator $\hat{\vec{V}}$ such that

$$\langle \hat{\vec{V}} \rangle = \frac{d}{dt} \langle \hat{\vec{Q}} \rangle .$$

For a state at time "t", $\Psi(t)$, this equation reads

$$\langle \Psi(t) | \hat{\vec{V}} | \Psi(t) \rangle = \frac{d}{dt} \langle \Psi(t) | \hat{\vec{Q}} | \Psi(t) \rangle$$

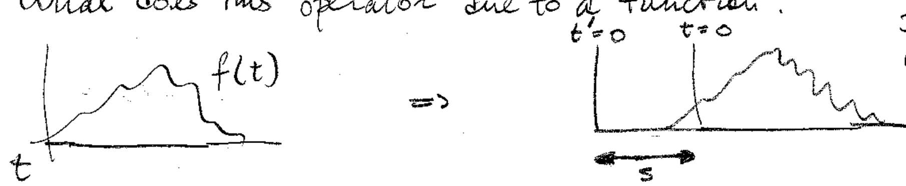
no time dependence to operator

$$= \left\{ \frac{d}{dt} \langle \Psi(t) | \right\} \hat{\vec{Q}} | \Psi(t) \rangle + \langle \Psi(t) | \hat{\vec{Q}} \left\{ \frac{d}{dt} | \Psi(t) \rangle \right\}$$

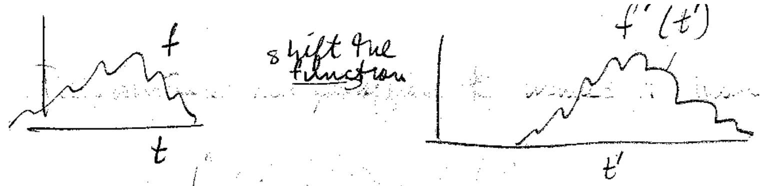
Now, we need to get an expression for $\frac{d}{dt} |\Psi(t)\rangle$. Let's go back to our time displacement operator:

$$e^{is\hat{H}} \quad \text{shifts time } t \text{ to } t' = t + s.$$

What does this operator due to a function?



If we shift t, then we need a new function f' so that $f'(t') = f(t)$



could use t' or t, its just a variable.

That is $f'(t') = f(t)$ or $f'(t+s) = f(t)$
or $f'(t) = f(t-s)$.
e.g. $f'(0) = f(-s)$

So, shifting t to $t+s$ shifts the function to its value at $t-s$:

$$\hat{U}(s) f(t) = f'(t) = f(t-s)$$

[this is the active transformation; shift the function to its new value]

So, $$e^{is\hat{H}} |\psi(t)\rangle = |\psi(t-s)\rangle.$$

Suppose $s = t$:
$$e^{it\hat{H}} |\psi(t)\rangle = |\psi(0)\rangle$$
or $$|\psi(t)\rangle = e^{-it\hat{H}} |\psi(0)\rangle.$$

Finally, $$\frac{d}{dt} |\psi(t)\rangle = \left\{ \frac{d}{dt} e^{-it\hat{H}} \right\} |\psi(0)\rangle$$
$$= -i\hat{H} \left\{ e^{-it\hat{H}} |\psi(0)\rangle \right\} = -i\hat{H} |\psi(t)\rangle.$$

Going back to our expression for the velocity:

$$\langle \psi(t) | \hat{\vec{V}} | \psi(t) \rangle = +i \langle \psi | \hat{H} \hat{\vec{Q}} | \psi \rangle - i \langle \psi | \hat{\vec{Q}} \hat{H} | \psi \rangle$$
$$= i \langle \psi | [\hat{H}, \hat{\vec{Q}}] | \psi \rangle$$

or $$\hat{\vec{V}} = i[\hat{H}, \hat{\vec{Q}}]$$

Of course, we still don't know what $\hat{H}$ is, but were making progress.

Now, we said that the displacement operator (in space or time) produces a new function evaluated at the old coordinate:

$$e^{is\hat{H}} |\psi(t)\rangle = |\psi'(t)\rangle .$$

and now

$$e^{-i\vec{a}\cdot\hat{\vec{P}}} |\vec{x}\rangle = |\vec{x}\rangle'$$

prime to denote new state vector

Now $|\vec{x}\rangle'$ is a function and the <u>function</u> is shifted with respect to the coordinate system:

$$|\vec{x}\rangle' = |\vec{x}+\vec{a}\rangle$$

not $\vec{x}-\vec{a}$; only $\vec{x}-\vec{a}$ if $\vec{x}$ is an <u>argument</u> of $f(\vec{x})$; here $\vec{x}$ denotes the function. $|\vec{x}\rangle$ is the original fn, $|\vec{x}+\vec{a}\rangle$ is the new function.

So $e^{-i\vec{a}\cdot\hat{\vec{P}}} |\vec{x}\rangle = |\vec{x}+\vec{a}\rangle$: the operator $e^{-i\vec{a}\cdot\vec{P}}$ shifts the state vector from $|x\rangle$ to $|x+a\rangle$.

What does $e^{-i\vec{a}\cdot\hat{\vec{P}}}$ do to the operator $\hat{Q}$? $\hat{\vec{P}}$ is the generator of translations

$$\hat{Q}' = U\hat{Q}U^{-1} = e^{-i\vec{a}\cdot\hat{\vec{P}}}\,\hat{Q}\,e^{+i\vec{a}\cdot\hat{\vec{P}}}$$
$$= (1 - i\vec{a}\cdot\hat{\vec{P}} \ldots)\,\hat{Q}\,(1 + i\vec{a}\cdot\hat{\vec{P}} \ldots)$$
$$= \hat{Q} + i[\hat{Q}, \vec{a}\cdot\hat{\vec{P}}] \ldots \qquad (1)$$

<u>But</u>, $\hat{Q}'|x\rangle' = x|x\rangle'$
$= x|x+a\rangle \quad (2)$

And, from way back $\hat{Q}|x\rangle = x|x\rangle$
$\Rightarrow \hat{Q}|x+a\rangle = (x+a)|x+a\rangle$.
$= x|x+a\rangle + a|x+a\rangle$. (3)

substitute

$\therefore \hat{Q}'|x\rangle' = \hat{Q}|x+a\rangle - a|x+a\rangle$ or $\hat{Q}' = \hat{Q} - \vec{a}\hat{I} \quad (4)$