Comparing ① with ④:

$$i\,[\vec{\hat{Q}}, \underbrace{\vec{a}\cdot\vec{\hat{P}}}_{\text{scalar}}] = -\vec{a}\,\hat{I} \quad \Rightarrow \quad i\,[\hat{Q}_\alpha, \vec{a}\cdot\vec{\hat{P}}] = -a_\alpha \hat{I}$$

These components must match.

$$\Rightarrow\; = i^2 a_\alpha \hat{I}.$$

Suppose $a_\alpha = 1 \quad a_\beta = a_\gamma = 0$ $\qquad a_\beta = 1 \;, a_\alpha = a_\gamma = 0$

$$[\hat{Q}_\alpha, \hat{P}_\alpha] = i\hat{I} \qquad [\hat{Q}_\alpha, \hat{P}_\beta] = 0$$

$$\Rightarrow \quad [\hat{Q}_\alpha, \hat{P}_\beta] = i\delta_{\alpha\beta}\,\hat{I}$$

This is beginning to look familiar, but we still haven't linked $\vec{\hat{P}}$ to momentum.

A similar argument to the above (which shows $\hat{P}$ generates shifts of coordinates) shows that $\hat{G}$ generates shifts in velocity.

$$\vec{\hat{V}}' = \vec{\hat{V}} - \vec{v}\,\hat{I}\;.$$

Now, we are trying to identify objects like $\hat{G}, \hat{P}, \hat{H}$ with $\hat{Q}, \hat{P}$ ...

$$[\hat{Q}_\alpha, \hat{P}_\beta] = i\delta_{\alpha\beta}\hat{I}$$

$$[\hat{G}_\alpha, \hat{P}_\beta] = i\delta_{\alpha\beta} M \hat{I}.$$

So try $\hat{G}_\alpha = M\hat{Q}_\alpha$ $\qquad \leftarrow$ Ballentine has a much longer discussion showing that this is unique.

Substitute this into $[\hat{G}_\alpha, \hat{H}] = i\hat{P}_\alpha$ (from summary)

$$\Rightarrow [\hat{Q}_\alpha, \hat{H}] = i\hat{P}_\alpha / M$$

But on page (3-14) we said that $V = i\,[H, Q] \Rightarrow [Q_\alpha, H] = iV_\alpha$

So try

$$\vec{\hat{P}} = M\vec{\hat{V}}$$

Using these forms for G and P, we can show that if $\hat{H}$ is of the form

$$\hat{H} = \frac{\vec{P}\cdot\vec{P}}{2M} + \varepsilon_0 \hat{I}$$

then $\hat{H}$ satisfies the appropriate commutation relations with G, P. Finally, we have

$$\vec{P} = M\vec{V}$$
$$H = \tfrac{1}{2} M\vec{V}\cdot\vec{V} + \varepsilon_0$$
$$\vec{J} = \vec{Q} \times M\vec{V} \qquad \text{(shown in Ballentine).}$$

We still don't know what M is, it just came out of a commutation relation. If we put

$$M = \text{mass}/\hbar \quad , \hbar \text{ being some constant,}$$

then we can rewrite all of the commutators in terms of physical quantities.

$$\frac{M}{\text{mass}} = \frac{P}{\text{momentum}} = \frac{H}{\text{energy}} = \frac{\vec{J}}{\text{angular momentum}} = \frac{1}{\hbar}.$$

The above equations were derived without reference to internal structure or spin. These more general situations are treated in Ballentine, pgs 61, 62. Ballentine also treats the situation in which there is an interaction with an external field. He shows that the general forms of $\hat{\vec{V}}$ and $\hat{H}$ become

$$\hat{\vec{V}} = \frac{\hat{\vec{P}}}{M} - \hat{\vec{A}}(\vec{Q})$$
$$\hat{H} = \frac{[(\hat{\vec{P}} - \hat{\vec{A}})]^2}{2M} + \hat{W}(\vec{Q})$$

units of $\hat{A}$ don't seem to agree.

where $\hat{\vec{A}}$ is a vector potential
and $\hat{W}$ is a scalar potential.

Equations of Motion

In the more traditional approach to QM, one defines the wave equation and then shows that the expectations obey Ehrenfest's Theorem. Here, we have used the classical correspondence to find the form of the generators. Let's now complete the equations of motion. We found that

$$\frac{d}{dt}|\psi(t)\rangle = \frac{-i}{\hbar}\hat{H}(t)|\psi(t)\rangle \qquad (1)$$

[We have now substituted Mass = ħM etc.].

We could also simply introduce a time-evolution operator $\hat{U}(t,t_0)$ which "evolves" the state from t_0 to t:

$$|\psi(t)\rangle = \hat{U}(t,t_0)|\psi(t_0)\rangle . \qquad (2)$$

The operator must ~~then~~ satisfy (from (1))

$$\frac{\partial}{\partial t}\hat{U}(t,t_0) = -\frac{i}{\hbar}\hat{H}(t)\hat{U}(t,t_0) \qquad (3)$$

with the obvious condition $\hat{U}(t_0,t_0)=1$. From (3),

$$\left.\begin{aligned}\hat{U}^{+}\frac{\partial}{\partial t}\hat{U} &= -\frac{i}{\hbar}\hat{U}^{+}\hat{H}\hat{U}\\ \left(\frac{\partial}{\partial t}U^{+}\right)\hat{U} &= +\frac{i}{\hbar}U^{+}H^{+}U\end{aligned}\right] \xrightarrow{\text{add}} \frac{\partial}{\partial t}(U^{+}U) = \frac{i}{\hbar}\left(U^{+}H^{+}U - U^{+}HU\right)$$

If $\hat{H}=\hat{H}^{+}$, as we expect, then the r.h.s. vanishes and

$$\frac{\partial}{\partial t}(U^{+}U)=0 \Rightarrow U^{+}U=1 \text{ for all time} \Rightarrow U \text{ is unitary.}$$

Further:

If $\hat{H}$ is time independent, solution of (3) is $\hat{U}(t,t_0)=e^{-i(t-t_0)\hat{H}/\hbar}$

If $\hat{H}$ is <u>not</u> time independent, no simple solution.

Let's apply this to the state operator ρ. For a pure state:

$$\rho(t) = |\psi(t)\rangle\langle\psi(t)| = \hat{U}(t,t_0)|\psi(t_0)\rangle\langle\psi(t_0)|\hat{U}^{\dagger}(t,t_0)$$
$$= \hat{U}(t,t_0)\,\hat{\rho}(t_0)\,\hat{U}^{\dagger}(t,t_0)$$

Further

$$\frac{d}{dt}\hat{\rho}(t) = \left\{\frac{\partial}{\partial t}\hat{U}(t,t_0)\right\}|\psi(t_0)\rangle\langle\psi(t_0)|\hat{U}^{\dagger}(t,t_0)$$
$$+ \hat{U}(t,t_0)|\psi(t_0)\rangle\langle\psi(t_0)|\left\{\frac{\partial}{\partial t}\hat{U}^{\dagger}(t,t_0)\right\}$$

$$= -\frac{i}{\hbar}\hat{H}\hat{U}|\psi\rangle\langle\psi|\hat{U}^{\dagger} + \frac{i}{\hbar}\hat{U}|\psi\rangle\langle\psi|\hat{U}^{\dagger}\underbrace{\hat{H}^{\dagger}}_{=H}$$

$$= -\frac{i}{\hbar}\left[\hat{H}\hat{\rho}(t) - \hat{\rho}(t)H\right]$$

$$= -\frac{i}{\hbar}\left[\hat{H},\hat{\rho}\right]$$

Although we did this for a pure state we assume that it should apply to a mixed state

The time dependence of the expectation of an observable is then

$$\langle R\rangle_t = \mathrm{Tr}\,(\rho(t)\hat{R}) \qquad (4)$$

This is referred to as the Schrodinger picture, in which $\hat{\rho}$ is time dependent but the operators $\hat{R}$ are not. But we could rearrange (4) to read

Heisenberg picture ρ_0, $R(t)$.

$$\langle R\rangle_t = \mathrm{Tr}\left(\hat{U}\hat{\rho}_0\hat{U}^{\dagger}R\right) = \mathrm{Tr}\,(\hat{\rho}_0\,\underbrace{\hat{U}^{\dagger}\hat{R}\hat{U}}_{\text{call this a time-dependent } \hat{R}(t)})$$

we use only the Schrodinger picture

The time rate of change of $\langle R\rangle_t$ is then

$\longrightarrow$ next page

$$\frac{d}{dt}\langle \mathcal{R}\rangle_t = \mathrm{Tr}\left(\frac{\partial\hat{\rho}}{\partial t}\hat{R} + \hat{\rho}\frac{\partial\hat{R}}{\partial t}\right)$$
$$= \mathrm{Tr}\left(-\frac{i}{\hbar}(\hat{H}\hat{\rho} - \hat{\rho}\hat{H})\hat{R} + \hat{\rho}\frac{\partial\hat{R}}{\partial t}\right)$$
$$= \mathrm{Tr}\left(-\frac{i}{\hbar}(\hat{\rho}\hat{R}\hat{H} - \hat{\rho}\hat{H}\hat{R}) + \hat{\rho}\frac{\partial\hat{R}}{\partial t}\right)$$
$$= \mathrm{Tr}\left\{\hat{\rho}(t)\left[\frac{i}{\hbar}[\hat{H},\hat{R}] + \frac{\partial\hat{R}}{\partial t}\right]\right\}$$

$\hat{R}$ may or may not be time-dependent.

<u>Symmetries and conservation Laws</u>

Suppose, finally, that we have some general variable $\hat{R}$. Then from

$$\frac{d}{dt}\langle \mathcal{R}\rangle_t = \mathrm{Tr}\left\{\hat{\rho}\left(\frac{i}{\hbar}[\hat{H},\hat{R}] + \frac{\partial\hat{R}}{\partial t}\right)\right\}$$

if $\hat{R}$ is independent of time and $[\hat{H},\hat{R}]=0$ then $\langle\mathcal{R}\rangle_t$ is conserved. There are other, similar expressions for invariances:

say $\hat{U}(s) = e^{i\hat{K}s}$

Then if $\hat{H}$ is invariant under U: $\hat{U}(s)\hat{H}\hat{U}^{-1}(s) = H$, we can show that

$$\mathbb{1} = (\mathbb{1} + i\hat{K}s \cdots)\hat{H}(\mathbb{1} - i\hat{K}s \cdots)$$
$$= \mathbb{1} + is\,\underline{[\hat{K},\hat{H}]} + \cdots$$

This must vanish if $\hat{H}$ is invariant under U.