Matroids with the Circuit Cover Property

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Abstract

We verify a conjecture of P. Seymour (*Europ. J. Combinatorics* **2**, p. 289) regarding circuits of a binary matroid. A *circuit cover* of a integer-weighted matroid (M, p) is a list of circuits of M such that each element e is in exactly $p(e)$ circuits from the list. We characterize those binary matroids for which two obvious necessary conditions for a weighting (M, p) to have a circuit cover are also sufficient.

Keywords: matroids, circuit covers, cycle covers, Hilbert base, sums of circuits, bond covers, cut covers

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1 Introduction

In this paper, we verify a conjecture of P. Seymour [13, (16.5)] which regards covering the elements of a binary matroid with circuits. We give a forbidden-minor characterization of those matroids which have a certain "Circuit Cover Property". The special case regarding graphic matroids, solved in [1], has had a number of implications for problems such as the Cycle Double Cover Conjecture, the Chinese Postman Problem, and Eulerian Decompositions [3, 6, 8, 17, 18, 19, 20]. We extend this result to the class of binary matroids by applying some fairly standard matroid decomposition techniques. This strategy involves verifying the conjecture for certain small matroids, and demonstrating that the relevant properties are preserved under matroid sums. The most involved single step (Lemma 5.4) requires checking that the bonds of a particular graph of order eight satisfy the conjecture. We assume familiarity with basic matroid theory such as in [15].

Let M be a binary matroid on ground set $E = E(M)$ and let $p : E \to \mathbb{Z}$. A *circuit cover* of the weighed matroid (M, p) is a list of circuits of M such that each element e is contained in exactly $p(e)$ circuits in the list. If (M, p) has a circuit cover, then the following three *admissibility conditions* must hold for any $e \in E$ and any cocircuit B.

1.1 $\quad p(e) \in \mathbb{Z}_+$

1.2 $\quad p(B) \equiv 0 \pmod 2$

1.3 $\quad$ *If* $e \in B$, *then* $p(e) \leq p(B - e)$.

(Here, $\mathbb{Z}_+$ denotes the non-negative integers. For $S \subseteq E$ we write $p(S)$ for $\sum_{e \in S} p(e)$, and $S - e$ for $S - \{e\}$.) The necessity of these conditions follows from the fact that, in a binary matroid, any circuit meets each cocircuit in an even number of elements. We say that (M, p) is *eulerian* if it satisfies 1.2 for all cocircuits B, (M, p) is *balanced* if it satisfies 1.3 for all pairs (B, e), and (M, p) is *admissible* if it satisfies 1.1, 1.2 and 1.3 for all pairs (B, e). A binary matroid M has the *circuit cover property* if (M, p) has a circuit cover for every admissible weight function p.

1.4 Theorem (Main Theorem) *Let M be a binary matroid. Then M has the circuit cover property if and only if M has no minor isomorphic to any of F_7^*, R_{10}, $M^*(K_5)$ or $M(P_{10})$.*

We describe here these matroids and some terminology. If M is a matroid then M^* denotes its dual. For any graph $G = (V, E)$, $M(G)$ denotes the matroid on $E(G)$ whose circuits are the edge sets of *polygons* (simple closed walks) in G. The dual matroid $M^*(G)$ has as circuits the edge sets of *bonds* (minimal edge cuts) in G. Matroids of the form $M(G)$ ($M^*(G)$) for some graph G are said to be *graphic* (*cographic*). The complete graph on n vertices is denoted K_n, complete bipartite graphs are denoted with $K_{s,t}$, and Petersen's graph is denoted P_{10}. The *Wagner graph* V_8 (sometimes called the *Möebius ladder* of order eight) is obtained from the polygon $v_0 v_1 \ldots v_7 v_0$ by adding four new edges of the form $v_i v_{i+4}$. The *Fano* matroid F_7 is the binary matroid represented over GF(2) by the seven non-zero binary 3-tuples. Thus the circuits of F_7^* are precisely the 4-arcs of the projective plane PG(2,2). We denote by R_{10} the unique

10-element regular matroid which is neither graphic nor cographic (see [13]). This matroid is conveniently represented by the edges of K_5, where $S \subseteq E(K_5)$ is a circuit of R_{10} if and only if S induces in K_5 either a polygon of length four or the complement of a polygon of length four.

As graphic matroids have no minors isomorphic to F_7^*, R_{10} or $M^*(K_5)$, the main theorem extends (and relies on) the following result which was proved in [1].

1.5 Corollary *A graphic matroid $M(G)$ has the circuit cover property if and only if G has no subgraph contractible to P_{10}.* □

A circuit cover of a weighted cographic matroid $(M^*(G), p)$ corresponds to a covering of $E(G)$ with bonds. Thus a graph G has the *bond cover property* if (G, p) has a bond cover for every $p : E(G) \to \mathbb{Z}_+$ such that (the edge set of) every polygon has even total weight and no edge has more than half the total weight of any polygon containing it. As none of F_7^*, R_{10} and $M(P_{10})$ is cographic, Theorem 1.4 implies the following.

1.6 Corollary *A graph has the bond cover property if and only if it has no subgraph contractible to K_5.*

□

2 Bad Minors

Let M be a matroid. For $S \subseteq E(M)$, the $(0,1)$-characteristic vector in $\mathbb{Q}^{E(M)}$ corresponding to S is denoted with χ^S. A weighted matroid (M, p) is *circuit minimal* if (M, p) is admissible, but $(M, p - \chi^C)$ is not admissible for any circuit C of M. If (M, p) is circuit minimal then (M, p) has no circuit cover, and M does not have the circuit cover property. A *k-circuit* is a circuit of cardinality k.

2.1 Lemma *None of F_7^*, R_{10}, $M^*(K_5)$ and $M(P_{10})$ has the circuit cover property.*

PROOF. For each of these matroids we describe a circuit-minimal weighting p.

F_7^*: Let C be a fixed 4-circuit of F_7^*. Put $p(e) = 1$ for all $e \in C$, and $p(e) = 2$ for the remaining 3 elements in F_7^*.

R_{10}: Let S be any 3-subset of elements not contained in any 4-circuit of R_{10}. Put $p(e) = 3$ for all $e \in S$, and $p(e) = 1$ for the remaining 7 elements in R_{10}.

$M^*(K_5)$: Let T be six edges in K_5 which induce a subgraph isomorphic with $K_{2,3}$. Put $p(e) = 1$ for all $e \in T$, and $p(e) = 2$ for the remaining four edges in K_5.

$M(P_{10})$: Let F be the edges in a fixed 1-factor of P_{10}. Put $p(e) = 2$ for all $e \in F$, and $p(e) = 1$ for the remaining 10 edges in P_{10}.

It is routine to verify that all four weighted matroids are circuit minimal. □

2.2 Remark There is an easy way to check that the first three of the above four weighted matroids (M,p) have no circuit cover. In each case, we define a function s on $E(M)$ via

$$s(e) = \begin{cases} 1 & \text{if } p(e) = 1, \\ -1 & \text{otherwise.} \end{cases}$$

One easily checks that s has negative inner product with p whereas $s(C)$ is non-negative for each circuit C of M. By Farkas' lemma p cannot be expressed as a linear combination of the vectors $\{\chi^C : C \text{ is a circuit in } M\}$ with nonnegative coefficients, whereas a circuit cover corresponds to such a linear combination with nonnegative integer coefficients. This argument fails for $M(P_{10})$ since the weighting described above is a nonnegative half-integral combination of circuits.

A *cycle* in a binary matroid M is any disjoint union of circuits in M. The cycles of M form a subspace of $\mathrm{GF}(2)^{E(M)}$, called the *cycle space*, under the symmetric difference operator "$\triangle$". Clearly (M,p) has a *cycle cover* if and only if it has a circuit cover. The terms *cocycle* and *cocycle space* are defined analogously. If (M,p) is a weighted matroid, then any minor M' of M, induces a *weighted minor* $(M', p\restriction_{E(M')})$, which we often denote by (M',p) where no confusion results. As a further abuse, we may write $e \in M$ and $p : M \to \mathbb{Z}$ instead of $e \in E(M)$ and $p : E(M) \to \mathbb{Z}$. We say that a cocycle D is *balanced* in (M,p) if $p(e) \leq p(D - e)$ for all $e \in D$.

2.3 Lemma *If a binary matroid M has the circuit cover property then any minor of M also has the circuit cover property.*

Proof. Assume M has the circuit cover property and let $f \in M$. We show that both $M \backslash f$ and M/f have the circuit cover property. First, suppose that $(M \backslash f, p)$ is admissible. We extend the definition of p to M by setting $p(f) := 0$. Then (M,p) is clearly admissible and thus has a circuit cover (C_i). Since $p(f) = 0$, (C_i) is also a circuit cover of $(M \backslash f)$, so $M \backslash f$ has the circuit cover property.

Now suppose that $(M/f, p)$ is admissible. We may assume that f is contained in some cocircuit of M, since otherwise f is a loop and $M/f = M \backslash f$. We extend p to $E(M)$ by setting

$$p(f) := \min\{p(B - f) : B \text{ is a cocircuit in } M \text{ containing } f\}.$$

Let B_0 be a cocircuit in M achieving this minimum. We claim that (M,p) is admissible.

Let B be any cocircuit in M. We may assume $f \in B$, for otherwise B is a cocircuit in M/f, whence 1.2 and 1.3 hold for B. Since $B \triangle B_0$ is a cocycle in M not containing f, $B \triangle B_0$ is a cocycle in M/f, so its total weight is even. We have, modulo 2, that $p(B \triangle B_0) \equiv p(B) + p(B_0)$, so $p(B) \equiv p(B_0) = 2p(f) \equiv 0$. Thus (M,p) is eulerian. Let $e \in B$. If $e \in B \cap B_0$, then we have $p(e) \leq p(B_0 - e) \leq p(B - e)$, by the choice of B_0. If $e \in B - B_0$, then $B \triangle B_0$ is a cocycle of M/f containing e, and thus $p(e) \leq p(B \triangle B_0 - e) \leq p(B - f) + p(B_0 - f) - p(e) = p(B - f) + p(f) - p(e) = p(B - e)$. Hence (M,p) is balanced, and thus admissible. Let (C_i) be a circuit cover of (M,p). Each $C_i - \{f\}$ is a cycle in M/f. Thus $(C_i - \{f\})$ is a cycle cover of $(M/f, p)$, and M/f has the circuit cover property.

□

3 Decomposition Theorems

We derive here a decomposition theorem for the class of matroids with which this paper is concerned. Although we use the terminology of Truemper [14], it suits our purposes to define matroid sums in terms of cycles and cocycles (as in Seymour [12]) rather than the matrix operations of Truemper. We refer the reader to Sections 8.5, 10.5 and 11.3 of [14], as well as to Section 12.4 of [15].

Let M_1, M_2 be binary matroids whose ground sets E_1, E_2 may intersect. We denote by $M_1 \triangle M_2$ the matroid on $E_1 \triangle E_2$ whose cycles are all subsets of $E_1 \triangle E_2$ of the form $C_1 \triangle C_2$, where C_i is a cycle in M_i, $i = 1, 2$. In particular, we define the following three *matroid sums*.

1. $M_1 \triangle M_2$ is a 1-*sum* of M_1 and M_2 if $E_1 \cap E_2 = \emptyset$.
2. $M_1 \triangle M_2$ is a 2-*sum* of M_1 and M_2 if $E_1 \cap E_2 = \{f\}$, where f is neither a loop nor a coloop of M_1 or M_2.
3. $M_1 \triangle M_2$ is a *Y-sum* of M_1 and M_2 if $|E_1 \cap E_2| = 3$, where $Z := E_1 \cap E_2$ is a cocircuit of size 3 in both M_1 and M_2, and Z contains no circuit of either M_1 or M_2.

A matroid sum of M_1 and M_2 is said to be *proper* if it contains proper minors isomorphic with M_1 and M_2. If M and M' are matroids, then an M' *minor* of M is a minor of M isomorphic with M'. We state two classical decomposition results. The first is due to Seymour (see [12, (11.3.16) and (11.3.19)]).

3.1 Lemma *Every binary matroid with no F_7^* minor may be constructed recursively by means of proper 1-sums, 2-sums and Y-sums, starting from copies of F_7, R_{10}, graphic, and cographic matroids.* □

We note that a Y-sum can not involve a copy of either R_{10} or F_7 since neither matroid has a cocircuit of cardinality three. The second decomposition result is essentially due to Wagner [16]. It is stated in dual form in [14, (10.5.15)]. A matroid is *planar* if it is both graphic and cographic.

3.2 Lemma *Every cographic matroid with no $M^*(K_5)$ minor may be constructed recursively by means of proper 1-sums, 2-sums and Y-sums starting from copies of $M^*(K_{3,3})$, $M^*(V_8)$, and planar matroids.* □

Seymour [12, (6.10)] observes that $M^*(K_{3,3})$, which is a minor of $M^*(V_8)$, may be dropped from the list of starting minors provided we do not require the sums to be proper. Indeed the following is easy to check.

3.3 Proposition *Every proper minor of $M^*(V_8)$ is either planar or is a Y-sum of two planar matroids.* □

3.4 Lemma *Every binary matroid with no F_7^*, R_{10}, $M^*(K_5)$ or $M(P_{10})$ minor may be constructed recursively by means of 1-sums, 2-sums and Y-sums starting from copies of F_7, $M^*(V_8)$, and graphic matroids containing no $M(P_{10})$ minor.*

Proof. Let M be a binary matroid with no minor isomorphic to F_7^*, R_{10}, $M^*(K_5)$ or $M(P_{10})$. We decompose M via Lemma 3.1, obtaining a list **L** of matroids. Since the sums in Lemma 3.1 are proper, each matroid in **L** is one of following.

- F_7
- cographic, with no $M^*(K_5)$-minor
- graphic, with no $M(P_{10})$-minor.

We have used here that $M(P_{10})$ is not cographic and $M^*(K_5)$ is not graphic. We further decompose each cographic matroid in **L** by applying Lemma 3.2. Finally we apply Proposition 3.3 to eliminate copies of $M^*(K_{3,3})$ and obtain our final decomposition. As planar matroids are graphic and have no $M(P_{10})$-minor, each matroid in the final decomposition is of the required type. □

4 Preservation under sums

We show here that the circuit cover property is preserved by 1-sums, 2-sums and Y-sums. We note that the operation that is dual to the Y-sum, called $\triangle$-sum by Truemper [14] and used by Seymour [12, 13], does not preserve the circuit cover property. For example, F_7 has the circuit cover property (Lemma 5.3), whereas any $\triangle$-sum of two copies of F_7 has an F_7^* minor. However, the proof of Lemma 4.2 below has a similar flavour to (7.2) and (7.3) of [13]. We first state an observation of Seymour [12, p. 319].

4.1 Lemma *Let M be a 1-sum, 2-sum or Y-sum of binary matroids M_1 and M_2. Then the cocycles of M are precisely the subsets of $E_1 \triangle E_2$ of the form $B_1 \triangle B_2$, where B_i is a cocycle of M_i, $i = 1, 2$.* □

4.2 Lemma *Suppose M is a 1-sum, 2-sum or Y-sum of binary matroids M_1 and M_2, where both M_1 and M_2 have the circuit cover property. Then M has the circuit cover property.*

Proof. We omit the 1-sum case as its proof is almost trivial. Let M be a 2-sum of M_1 and M_2 where $M_1 \cap M_2 = \{f\}$ and where each M_i has the circuit cover property. We proceed by constructing an admissible weighting p_i of M_i, $i = 1, 2$. Let $i \in \{1, 2\}$. Since f is not a loop in M_i, some cocircuit in M_i contains f. We choose such a cocircuit D_i which minimizes $p(D_i - \{f\})$, and let n_i denote this minimum. We assume without loss of generality that $n_1 \leq n_2$. For $i = 1, 2$ we define $p_i : M_i \to \mathbb{Z}$ by

$$p_i(e) = \begin{cases} n_1 & \text{if } e = f \\ p(e) & \text{otherwise.} \end{cases}$$

From this definition we immediately have

$$p_1(e) \leq p_1(D_1 - \{e\}), \text{ for all } e \in D_1. \tag{1}$$

We now show that each (M_i, p_i) is admissible. Let $i \in \{1, 2\}$. By Lemma 4.1 each cocircuit D in M_i not containing f is a cocycle in M. Since (M, p) is admissible and since p and p_i coincide on D, D is balanced and eulerian in (M_i, p_i). We assume now that D is a cocircuit in M_i containing f. By Lemma 4.1, $D \triangle D_1$ is a cocycle of M, so $D \triangle D_1$ is balanced and has even total weight in (M, p). Since $p_1(D_1)$ is even and

$p(D \triangle D_1) \equiv p_i(D) + p_1(D_1) \pmod 2$, it follows that $p_i(D)$ is even. Let $e \in D$. If $e \in D \cap D_1$, then by (1) and the choice of D_1

$$p_i(e) = p_1(e) \leq p_1(D_1 - \{e\}) \leq p_i(D - \{e\}).$$

If $e \in D - D_1$, then $e \in D_1 \triangle D$ and

$$p_i(e) = p(e) \leq p(D \triangle D_1 - \{e\}) \leq p(D - \{f\}) + p(D_1 - \{f\}) - p(e) = p_i(D - \{f\}) + p_i(f) - p_i(e) = p_i(D - \{e\}).$$

Thus (M_i, p_i) is admissible and, by hypothesis, has a circuit cover $\mathbf{L}_i$, $i = 1, 2$. We form a cycle cover of (M, p) by "pairing off" the circuits in $\mathbf{L}_1$ which contain f with those in $\mathbf{L}_2$ which contain f. More precisely, we let $\mathbf{L}_i'$ be the sublist of $\mathbf{L}_i$ consisting of the n_1 circuits which contain f, $i = 1, 2$, and let $g : \mathbf{L}_1' \to \mathbf{L}_2'$ be any bijection. The list $(\mathbf{L}_1 - \mathbf{L}_1') \cup (\mathbf{L}_2 - \mathbf{L}_2') \cup (C \triangle g(C) : C \in \mathbf{L}_1')$ is a cycle cover of (M, p). Thus M has the circuit cover property.

We assume now that M is a Y-sum of M_1 and M_2 where each M_i has the circuit cover property. Let $M_1 \cap M_2 = Z = \{e_1, e_2, e_3\}$ where Z is a cocircuit in both M_1 and M_2. Let p be an admissible weighting of M. Let $i \in \{1, 2\}$ and $j \in \{1, 2, 3\}$. Since Z contains no circuit of M_i, e_j is not a loop in $M_i/(Z - \{e_j\})$ and thus there is a cocircuit D in M_i such that $D \cap Z = \{e_j\}$. Let d_{ij} be the minimum of $p(D - e_j)$ over all such cocircuits D, and let D_{ij} be a cocircuit attaining this minimum. For $j = 1, 2, 3$, we define $n_j := \min\{d_{1j}, d_{2j}\}$, and let D_j be a cocircuit in $\{D_{1j}, D_{2j}\}$ with $p(D_j - p) = n_j$. Let $n := n_1 + n_2 + n_3$. For $i = 1, 2$ we define the weighting $p_i : M_i \to \mathbb{Z}$ by

$$p_i(e) = \begin{cases} \min\{n_j, n - n_j\} & \text{if } e = e_j, \text{ for some } j \in \{1,2,3\} \\ p(e) & \text{otherwise.} \end{cases}$$

We now show each (M_i, p_i) is admissible. Let $i \in \{1, 2\}$. Let D be any cocircuit of M_i. We have four cases depending on $|Z \cap D|$.

Case $|Z \cap D| = 3$: Here $D = Z$. By construction of p_i, Z is balanced in (M_i, p_i). Its weight, $p_i(Z)$, equals either n or $2n - 2n_j$ for some $j \in \{1, 2, 3\}$, and thus Z is eulerian provided that n is even. By Lemma 4.1, $D_1 \triangle D_2 \triangle D_3 \triangle Z$ is a cocycle of M, and so $n = n_1 + n_2 + n_3 = p(D_1) + p(D_2) + p(D_3) - p(Z) \equiv p(D_1 \triangle D_2 \triangle D_3 \triangle Z) \equiv 0 \pmod 2$, as required.

Case $|Z \cap D| = 0$: Here Z is a cocycle of M, and is thus balanced and eulerian in (M_i, p_i).

Case $|Z \cap D| = 1$: By symmetry we may assume $Z \cap D = \{e_1\}$. If $p_i(e_1) = n_1$, then by the same argument as in the 2-sum case, D is eulerian and balanced in (M_i, p_i). We assume that $p_i(e_1) = n_2 + n_3 < n_1$, $p_i(e_2) = n_2$, and $p_i(e_3) = n_3$. We claim that neither M_1 nor M_2 contains both D_2 and D_3. Otherwise, $D_2 \triangle D_3 \triangle D$ would be a cocycle in M_1 or M_2 such that $(D_2 \triangle D_3 \triangle D) \cap Z = \{e_1\}$. This cocycle contains a cocircuit D_1' with $D_1' \cap Z = \{e_1\}$. For $k = 1, 2$ we have

$$p_k(D_1' - \{e_1\}) \leq p_k(D_2 - \{e_2\}) + p_k(D_3 - \{e_3\}) = n_2 + n_3 < n_1$$

contradicting the minimality of n_1 and proving our claim. Hence exactly one of D_2, D_3, say D_2, belongs to M_i. Now $D_3' := D_2 \triangle D \triangle Z$ is a cocycle of M_i with $D_3' \cap Z = \{e_3\}$. Since $p_i(D_3')$, $p_i(D_2)$ and $p_i(Z)$

are even, and $p_i(D_3') \equiv p_i(D_2) + p_i(D) + p_i(Z) \pmod 2$, $p_i(D)$ is even. We now show D is balanced in (M_i, p_i). Let $e \in D$. If $e \in D \cap D_2$, then

$$\begin{aligned} p_i(e) &\leq p_i(D_2 - \{e\}) = p_i(D_2 - \{e_2\}) + p(e_2) - p_i(e) \\ &= n_2 + n_2 - p_i(e) \leq 2(n_2 + n_3) - p_i(e) = 2p_i(e_1) - p_i(e) \\ &\leq p_i(D - \{e_1\}) + p_i(e_1) - p_i(e) = p_i(D - \{e\}). \end{aligned}$$

If $e \in D - D_2$, then $e \in D_3' = D_2 \triangle D \triangle Z$. Since $D_3' \cap Z = \{e_3\}$ and $p(e_3) = n_3$, D_3' is balanced and eulerian in (M_i, p_i) as in the second sentence of this case. Therefore

$$\begin{aligned} p_i(e) &\leq p_i(D_2 \triangle D \triangle Z - \{e\}) \leq p_i(D - \{e_1\}) + p(D_2 - \{e_2\}) + p_i(e_3) - p_i(e) \\ &= p_i(D - \{e_1\}) + p_i(e_1) - p_i(e) = p_i(D - \{e\}). \end{aligned}$$

Case $|Z \cap D| = 2$: We may assume that $D \cap Z = \{e_1, e_2\}$ so that $D \triangle Z$ is a cocycle of M_i satisfying $(D \triangle Z) \cap Z = \{e_3\}$. As in the previous case, $p_i(D \triangle Z)$ is even. Since $p_i(Z)$ is even, $p_i(D)$ is even. We now show that D is balanced in (M_i, p_i) Let $e \in D$. If $e \in D - \{e_1, e_2\}$, then $e \in D \triangle Z$. By the previous case $D \triangle Z$ is balanced in (M_i, p_i) so

$$p_i(e) \leq p_i(D \triangle Z - \{e\}) = p_i(D) - p_i(e_1) - p_i(e_2) + p_i(e_3) - p_i(e) \leq p_i(D - \{e\}).$$

Lastly, we have

$$p_i(e_1) \leq p_i(e_2) + p_i(e_3) \leq p_i(e_2) - p_i(D - \{e_1, e_2\}) = p_i(D - \{e_1\})$$

and similarly $p_i(e_2) \leq p_i(D - \{e_2\})$.

We have shown (M_i, p_i) is admissible, $i = 1, 2$. By hypothesis (M_i, p_i) has a circuit cover $\mathbf{L}_i$, $i = 1, 2$. Each circuit in $\mathbf{L}_i$ which intersects Z in 0 or 2 elements. For $j = 1, 2, 3$ we denote by $\mathbf{L}_i^j$ those circuits in $\mathbf{L}_i$ containing $Z - \{e_j\}$. For any partition $\{j\} \cup \{k, \ell\}$ of $\{1, 2, 3\}$ we have $|\mathbf{L}_1^k| + |\mathbf{L}_1^\ell| = p_1(e_j) = p_2(e_j) = |\mathbf{L}_2^k| + |\mathbf{L}_2^\ell|$, and thus $|\mathbf{L}_1^j| = |\mathbf{L}_2^j| = (p_1(e_k) + p_1(e_\ell) - p_1(e_j))/2$. Similarly to the 2-sum case, the circuits in in $\mathbf{L}_1^j$ may be "paired off" with those in $\mathbf{L}_2^j$, $j = 1, 2, 3$ in an obvious way to yield a circuit cover of (M, p). Thus (M, p) has the circuit cover property. □

5 Good Matroids

It remains to show that each of the building blocks of the decomposition of Lemma 3.4 has the circuit cover property. The first is due to Alspach, Goddyn and Zhang [1].

5.1 Lemma *Graphic matroids containing no $M(P_{10})$ minor have the circuit cover property.*

We define a partial order on the set of weightings of a matroid M. For $p, q : M \to \mathbb{Z}$ we write $p \preceq q$ if $p(e) \leq q(e)$ for each $e \in M$. A weighting (M, p) is *positive* if $p(e) \geq 1$ for all $e \in M$. Let B be a cocircuit in M and let $e \in B$. We define the *slack* of (B, e) in (M, p) by

$$\mathrm{sl}(B, e) = \mathrm{sl}_p(B, e) := p(B - e) - p(e).$$

Thus if (M, p) is admissible, then $\mathrm{sl}(B, e)$ is a nonnegative even integer. The following is convenient for showing that a matroid has the circuit cover property.

5.2 Lemma *Let M be a matroid such that $M\backslash e$ has the circuit cover property for all $e \in M$. Suppose further that, for any admissible positive weighting (M, p), there exists a circuit C such that for any cocircuit B and $e \in B$,*

$$\mathrm{sl}(B, e) \geq \begin{cases} |C \cap B| & \text{if } e \notin C \\ |C \cap B| - 2 & \text{if } e \in C. \end{cases} \tag{2}$$

Then M has the circuit cover property.

Proof. Suppose M satisfies the hypothesis, but does not have the circuit cover property. Let (M, p) be a $\preceq$-minimal admissible weighting which has no circuit cover. If $p(e) = 0$ for some $e \in M$, then the restriction $(M\backslash e, p)$ is admissible. By hypothesis $(M\backslash e, p)$ has a circuit cover. This circuit cover is also a circuit cover of (M, p), a contradiction. Thus p is positive, whence there exists a circuit C satisfying (2) for every cocircuit B and $e \in B$. Let $p' := p - \chi^C$. Since (M, p) is eulerian and positive, (M, p') is eulerian and nonnegative valued. For any $S \subseteq E(M)$ we have $p'(S) = p(S) - |C \cap S|$, so (2) is equivalent to the statement $p'(e) \leq p'(B - e)$. Thus (M, p') is admissible and, by minimality of p, (M, p') has a circuit cover. Adjoining C to this circuit cover yields a circuit cover of (M, p), a contradiction. Thus M has the circuit cover property. □

5.3 Lemma *The matroid F_7 has the circuit cover property.*

Proof. We check that F_7 satisfies the hypothesis of Lemma 5.2. For any $e \in F_7$, $F_7\backslash e \cong M(K_4)$ which has the circuit cover property by Lemma 5.1. Let (F_7, p) be admissible and positive. Let $a, b \in F_7$ be such that $p(e) \leq p(b) \leq p(a)$ for all $e \in E - \{a, b\}$, and let C be the unique circuit of cardinality 3 containing a and b. Let B be any cocircuit of F_7 and let $e \in B$. As $|C \cap B|$ is even and $|C| = 3$, $C \cap B$ contains either 0 or 2 elements. Since $\mathrm{sl}(B, e) \geq 0$, (2) holds unless $e \notin C$ and $|C \cap (B - e)| = 2$. In this case e is different from a and b. Every cocircuit of F_7 has cardinality 4 so $B - e$ contains 3 elements of positive weight. Since $|C - B| = 1$, one of these 3 elements is either a or b. We have $\mathrm{sl}(B, e) = p(B - e) - p(e) \geq (p(b) + 2) - p(e) \geq 2$ and (2) holds. Thus F_7 has the circuit cover property. □

The following lemma completes the proof of the main theorem. It is the most involved single step in its proof. The details are supplied in the next section.

5.4 Lemma *The matroid $M^*(V_8)$ has the circuit cover property.*

6 Bond covers of V_8

To minimize confusion we use a separate terminology for graphs. A *cycle* in a graph G is the edge set of a simple closed walk in G. An *edge cut* is the set $\delta(X)$ of edges with exactly one endpoint in X for some $X \subseteq V(G)$. A *bond* is an minimal nonempty edge cut. We say that G has the *bond cover property* if $M^*(G)$

has the circuit cover property. The graph V_8 is obtained from the polygon $v_0e_0v_1e_1\cdots e_6v_7e_7$ by adding the edges e_{04}, e_{15}, e_{26}, e_{37}, where each e_{ij} has endpoints v_i and v_j. Each e_i is called a *rim edge* whereas each e_{ij} is called a *spoke*. The automorphism group of V_8 is the dihedral group of order 16. Our aim is to show that V_8 has the bond cover property.

If G is a plane graph then G has the bond cover property if and only if the polygon matroid of its plane dual G^* has the the circuit cover property. As planar graphs do not have P_{10} as a minor, Lemma 2.3 implies the following.

6.1 Lemma *Every planar graph has the bond cover property.* □

Applying Lemmas 3.3 and 4.2, we have the following.

6.2 Lemma *All proper minors of V_8 have the bond cover property.* □

As in Section 5, we define

$$\mathrm{sl}(C,e) := p(C-e) - p(e)$$

for any cycle C and $e \in C$. We say that an edge weighted graph (G,p) is *bond admissible* if $(M^*(G),p)$ is admissible. Thus $p: E(G) \to \mathbb{Z}_+$ is bond admissible if for each cycle C and $e \in C$, $\mathrm{sl}(C,e)$ is a non-negative even integer. A cycle C in (G,p) is *balanced* if $\mathrm{sl}(C,e) \geq 0$ for all $e \in C$, and is *tight* if $\mathrm{sl}(C,e) = 0$ for some $e \in C$. If $\mathrm{sl}(C,e) = 0$, then e is called a *leader* of C. If G is simple and p is positive and bond admissible, then each tight cycle in (G,p) has a unique leader. A *chord* of a cycle C is an edge in $E(G) - C$ such that both its endpoints are incident with an edge in C. We recall that the symmetric difference of two cycles in G is an edge-disjoint union of cycles in G.

6.3 Lemma *Let p be a positive weighting of a graph G. Let C, C' be cycles and e, f be edges in G such that $e \in C - C'$ and $f \in C \cap C'$. Let D be a cycle in $C \triangle C'$ containing e. Then $\mathrm{sl}(D,e) \leq \mathrm{sl}(C,e) + \mathrm{sl}(C',f)$ with equality if and only if $D = C \triangle C'$ and f is a chord of D.*

Proof. Let $a = \mathrm{sl}(C,e)$ and $b = \mathrm{sl}(C',f)$. We have $p(C\triangle C' - e) = p(C-e-f) + p(C'-f) - 2p(C \cap C' - f)$ and so

$$\begin{aligned} p(e) &= p(C-e) - a = p(C-e-f) - a + p(f) \\ &= p(C-e-f) - a + p(C'-f) - b \\ &= p(C \triangle C' - e) + 2p(C \cap C' - f) - a - b \\ &\geq p(D-e) - a - b. \end{aligned}$$

Since p is positive, equality holds if and only if $D = C \triangle C'$, and $C \cap C' = \{f\}$. □

6.4 Corollary *Let C be a cycle in G, $e \in C$ and let f be a chord of C. Let C_e be the cycle in $C \cup \{f\}$ containing both e and f, and let C_f be the cycle in $C \cup \{f\} - \{e\}$. Then for any edge-weighting p we have $\mathrm{sl}(C,e) = \mathrm{sl}(C_e,e) + \mathrm{sl}(C_f,f)$.* □

For any edge weighting (V_8, p) we define the *set of leaders*

$$L = L(p) := \{e \in E(V_8) : e \text{ is the leader of some tight cycle in } (V_8, p)\}.$$

6.5 Lemma *Let (V_8, p) be positive and bond admissible. If $|C \cap L| \geq 2$ for some tight cycle C, then (V_8, p) has a bond cover.*

PROOF. Let $e, f \in C \cap L$ and assume that e is the (unique) leader of C. Then f is the leader of some tight cycle C' different from C. Since $p(f) < p(e)$, we have $e \notin C'$. Let D be the cycle in $C \triangle C'$ containing e. As D is a balanced in (V_8, p), Lemma 6.3 gives $0 \leq \mathrm{sl}(D, e) \leq \mathrm{sl}(C, e) + \mathrm{sl}(C', f) = 0$ so $D = C \triangle C'$ is a tight cycle and f is a chord of D.

Let x, y be the endpoints of f. Since every cycle in $V_8 \backslash f$ is a cycle in V_8, $(V_8 \backslash f, p)$ is bond admissible. By Lemma 6.2, $(V_8 \backslash f, p)$ has a bond cover $\mathbf{L}'$. Let $\mathbf{L}''$ be the sublist consisting of those bonds $\delta(X)$ in $\mathbf{L}'$ for which $|X \cap \{x, y\}| = 1$. Let $\mathbf{L}$ be the list of bonds in V_8 obtained from $\mathbf{L}'$ by adding the edge $f = xy$ to those bonds in $\mathbf{L}''$. Since D is tight, each bond in $\mathbf{L}'$ which contains e contains exactly one other edge in D, so the number of bonds in $\mathbf{L}''$ is exactly $p(C' - f) = p(f)$. Thus $\mathbf{L}$ is a bond cover of (V_8, p). □

We say that a bond B in (V_8, p) is *removable* if $(V_8, p - \chi^B)$ is bond admissible. In particular, any bond in a bond cover of (V_8, p) is removable. As in Lemma 5.2, if p is positive, then a bond B is removable if and only if for any cycle C and $e \in C$,

$$\mathrm{sl}(C, e) \geq \begin{cases} |B \cap C| & \text{if } e \notin B \\ |B \cap C| - 2 & \text{if } e \in B. \end{cases}$$

We derive a sufficient condition, depending only on the leaders $L(p)$, for a bond to be removable in (V_8, p). A *k-cycle* is a cycle of cardinality k.

6.6 Lemma *Let (V_8, p) be positive and bond admissible. Then a bond B is removable provided all of the following hold.*

1. *$L \subseteq B$*
2. *For every 4-cycle C contained in B, we have $|C \cap L| \geq 2$.*
3. *For every 5-cycle C such that $|C \cap B| = 4$, we have $|C \cap L| \geq 2$ and there exists $f \in C \cap L$ such that every chordless cycle C' containing f satisfies $|C' \cap (C \cup L)| \geq 2$.*

PROOF. Suppose that B is not removable, but that *1., 2., and 3.* hold. Let $p' = p - \chi^B$. Then some cycle is unbalanced in (V_8, p'). Applying Corollary 6.4, there exists an unbalanced cycle C in (V_8, p') which is chordless. Since every cycle in V_8 having cardinality greater than five has a chord, we have $4 \leq |C| \leq 5$. Since C is balanced in (V_8, p) but not in (V_8, p'), $|B \cap C|$ equals either 2 or 4. If $|B \cap C| = 2$, then C is tight in (V_8, p) and B does not contain the leader of C, contradicting *1.* Thus we have $4 = |C \cap B| \leq |C| \leq 5$. By *2.* and *3.*, $|C \cap L| \geq 2$. If C were tight in (V_8, p), then (V_8, p) would have a removable bond by Lemma 6.5. Thus C is not tight in (V_8, p) whereas C is unbalanced in (V_8, p'). Since $|C| \leq 5$ this can happen only if $|C| = 5$ and, for some $e \in C$, $C - B = \{e\}$ and $\mathrm{sl}_p(C, e) = 2$. Let $f \in C \cap L$ be the edge specified in condition *3.* Applying Corollary 6.4, f is a leader of some chordless tight cycle C'. Since $f \in C' \cap C \cap L$, condition *3.* implies that either $|C' \cap C| \geq 2$ or $|C' \cap L| \geq 2$. Applying Lemma 6.5 to C', we may assume $|C' \cap L| = 1$ and thus $|C' \cap C| \geq 2$. If $e \in C'$, then $p(f) = p(C' - f) \geq p(e) = p(C - e) - 2 \geq p(f) - 2$ which is absurd. Therefore $e \in C - C'$ and $f \in C \cap C'$. Let D be the cycle in $C \triangle C'$ which contains e. Applying Lemma 6.3 to (V_8, p), we have $\mathrm{sl}(D, e) \leq \mathrm{sl}(C, e) + \mathrm{sl}(C', f) = 2 + 0$. Since $|C' \cap C| \geq 2$, we do not have equality here, whence $\mathrm{sl}(D, e) = 0$. Thus $e \in L - B$, contradicting *1.* □

A *matching* is a set of edges such that no two are adjacent.

6.7 Lemma *Let (V_8, p) be positive and bond admissible. If L is not a matching of cardinality at least two, then V_8 has a removable bond.*

Proof. Let $v \in V(V_8)$. Every cycle intersects the star-bond $\delta(v)$ in at most two edges. Therefore if $\delta(v)$ is not removable, then there exists a tight cycle C in (V_8, p) which intersects $\delta(v)$ but whose leader is not contained in $\delta(v)$. Since v is arbitrary, we have $|L| \geq 2$. Furthermore, if v is incident with at least two edges in L, then one of these two edges is in C, since v has degree three. Thus $|C \cap L| \geq 2$ and by Lemma 6.5, (V_8, p) has a bond cover. □

We now complete the proof of Theorem 5.4. By Lemmas 5.2 and 6.2 it suffices to show the existence of a removable bond in any positive, bond admissible weighting (V_8, p). By Lemma 6.7 we may assume that $L(p)$ is a matching of cardinality at least two. For $S \subseteq E(V_8)$ we denote by $[S]$ the orbit of S under the group of automorphisms of V_8. Referring to Figure 1, we claim that for any matching L in V_8 with $|L| \geq 2$, there is a bond B in either $[B_1]$, $[B_2]$, $[B_3]$ or $[B_4]$ satisfying conditions *1., 2.,* and *3.* of Lemma 6.6. We note that condition *2.* is vacuously true for B_1, B_3 and B_4, and that condition *3.* is vacuously true for B_1, B_2 and B_3.

If L contains no rim edges and $|L| \leq 3$, then some bond in $[B_1]$ contains L, and thus satisfies *1., 2.,* and *3.* If L consists of all four spokes in V_8, then B_2 satisfies the three conditions. If L contains exactly one rim edge then, since L is a matching, some bond in $[B_1]$ contains L and is therefore removable. If L contains exactly two rim edges and these two edges are at distance 3 (4) along the rim of V_8, then some bond in $[B_1]$ ($[B_2]$) again satisfies the three conditions, and hence is removable. If L contains exactly two edges at distance 2 along the rim of V_8 then, since L is a matching, $|L| = 2$ and a bond in $[B_3]$ contains L and thus satisfies the three conditions. If L contains 3 or 4 rim edges, then L contains no spokes since L is a matching. If $L \subseteq B_3$, then B_3 is removable so we may assume that $L = \{e_0, e_2, e_5\}$. We claim that B_4 is removable in this case. In this case condition *3.* must be checked with $C = \{e_2, e_3, e_4, e_5, e_{26}\}$; here e_5 serves as the required edge f. In all cases, there is a removable bond in V_8, and thus V_8 has the bond cover property.

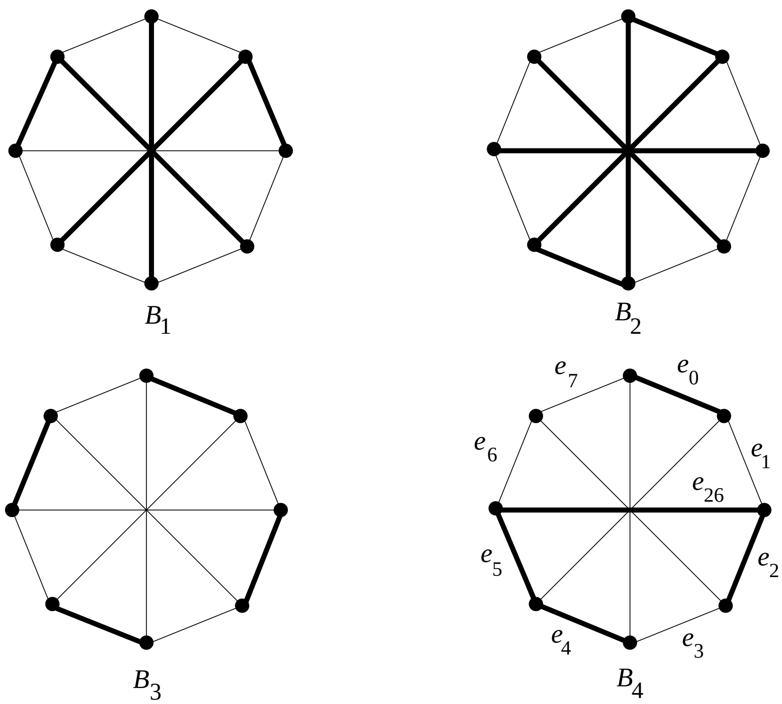

Figure 1: Four bonds in V_8

7 Remarks

Remark 2.2 provides a good reason that none of F_7^*, R_{10}, $M^*(K_5)$ has the circuit cover property; each of these matroids has an admissible weighting which does not even have a "fractional" circuit cover. This suggests that the admissibility conditions, 1.1, 1.2, 1.3, should be replaced with stronger ones. Let $\mathcal{C} = \mathcal{C}(M)$ denote the set of circuits in matroid M. Let $\mathbb{Q}_+$ denote the non-negative rational numbers. For any matroid M we define the *lattice*, the *cone* and the *integer cone* of circuits in M by

$$\begin{aligned}\mathbb{Z}(\mathcal{C}) &:= \{\sum_{C\in\mathcal{C}} \alpha_C \chi^C : \alpha_C \in \mathbb{Z} \text{ for all } C \in \mathcal{C}\}\\ \mathbb{Q}_+(\mathcal{C}) &:= \{\sum_{C\in\mathcal{C}} \alpha_C \chi^C : \alpha_C \in \mathbb{Q}_+ \text{ for all } C \in \mathcal{C}\}\\ \mathbb{Z}_+(\mathcal{C}) &:= \{\sum_{C\in\mathcal{C}} \alpha_C \chi^C : \alpha_C \in \mathbb{Z}_+ \text{ for all } C \in \mathcal{C}\}\end{aligned}$$

Thus (M,p) has a circuit cover if and only if $p \in \mathbb{Z}_+(\mathcal{C})$. For any matroid,

$$\mathbb{Z}_+(\mathcal{C}) \subseteq \mathbb{Q}_+(\mathcal{C}) \cap \mathbb{Z}(\mathcal{C}).$$

If equality holds here, then we say that $\mathcal{C}$ forms a *Hilbert base*. Let $\mathcal{H}$ denote the class of matroids whose circuits form a Hilbert base. The following problem is raised in [5].

7.1 Problem *Characterize the matroids in* $\mathcal{H}$.

It is known (see [5, 10]) that for any binary matroid M with no F_7^* minor, a weighting $p : M \to \mathbb{Z}$ belongs to $\mathbb{Z}(M)$ if and only if 1.2 holds for all cocircuits B and 1.3 holds for all cocircuits B with $|B| \leq 2$. Seymour [13] showed that, for all binary matroids with no F_7^*, R_{10} or $M^*(K_5)$ minor, a weighting $p : M \to \mathbb{Q}_+$ belongs to $\mathbb{Q}_+(M)$ if and only if 1.3 holds for all cocircuits B. Thus our main theorem partially answers the above problem.

7.2 Corollary *Let M be a binary matroid with no F_7^*, R_{10} or $M^*(K_5)$ minor. Then $M \in \mathcal{H}$ if and only if M has no $M(P_{10})$ minor.* □

If M contains a F_7^*, R_{10} or $M^*(K_5)$ minor, then our main theorem is no longer relevant since the cone of circuits of M is strictly contained in cone of non-negative weights p satisfying 1.3. Indeed, it is $\mathcal{NP}$-hard to determine whether $p \in \mathbb{Q}_+(M)$, even if M is cographic [7]. Some further progress has been made on Problem 7.1. For example, the dual of any projective geometry $\mathrm{PG}(n, q)$ (including $F_7^* = \mathrm{PG}^*(2, 2)$) belongs to $\mathcal{H}$ since its circuits are linearly independent in $\mathbb{Q}^E$. It follows from matroid partition theory (see [2]) that the bases of a matroid form a Hilbert base, and hence all uniform matroids belong to $\mathcal{H}$. M. Laurent [9] has shown that $\mathcal{H}$ contains every proper minor of $M^*(K_6)$ whereas no cographic matroid containing an $M^*(K_6)$ minor is in $\mathcal{H}$. As far we know, the following problem is open.

7.3 Problem *Is the class of cographic matroids in $\mathcal{H}$ closed under taking minors?*

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References

[1] B. Alspach, L. Goddyn and C-Q Zhang, Graphs with the circuit cover property, *Trans. Amer. Math. Soc.* **344** (1994), 131–154.

[2] W. Cook, J. Fonlupt, A. Schrijver An integer analogue of Carathéodory's Theorem, *J. Comb. Theory Ser. B* **40** (1986), 63–70.

[3] G. Fan and C-Q Zhang, Circuit Decompositions of Eulerian Graphs, *J. Comb. Theory Ser. B*, submitted.

[4] X-D Fu, Matroids with the circuit cover property, Masters thesis, Department of Mathematics and Statistics, Simon Fraser University, September, 1991.

[5] L. Goddyn, Cones, lattices and Hilbert bases of circuits and perfect matchings. *in* "Graph Structure Theory", N. Robertson, P. Seymour, Eds., *Contemporary Mathematics* **147**, pp. 419–440. American Mathematical Society, 1993.

[6] L. Goddyn, J. van den Heuvel, S. McGuinness, Removable circuits in multigraphs, in preparation.

[7] I. Holyer, The NP-completeness of edge-coloring, *SIAM J. Comput.* **10** (1981), 718–720.

[8] B. Jackson, On circuit covers, circuit decompositions and Euler tours of graphs, *in* "Surveys in Combinatorics," (Keele Ed.), *London Math. Soc. Lecture Note Ser.* **187**, pp. 191–210 Cambridge Univ. Press, Cambridge, 1993.

[9] M. Laurent, Hilbert bases of cuts, Research report *LIENS - 93 - 9*, Laboratoire d'Informatique, URA 1327 du CNRS, Département de Mathématiques et d'Informatique, Ecole Normale Supérieure, April 1993.

[10] L. Lovász, Á. Seress, The cocycle lattice of binary matroids. *Europ. J. Combinatorics* **14** (1993), 241–250.

[11] P. D. Seymour, Sums of circuits, *in* "Graph Theory and Related Topics", (J. A. Bondy and U. S. R. Murty, Eds.), pp. 341–355, Academic Press, New York/Berlin, 1979.

[12] P. D. Seymour, Decomposition of regular matroids, *J. Combin. Theory Ser. B* **28** (1980), 305–359.

[13] P. D. Seymour, Matroids and multicommodity flows, *Europ. J. Combinatorics* **2** (1981), 257–290.

[14] K. Truemper, *Matroid Decomposition*, Academic Press, Boston (1992).

[15] J. G. Oxley, *Matroid Theory*, Oxford University Press, New York (1992).

[16] K. Wagner, Uber eine Eigenschaft der evenen Komplexe, *Math. Ann.* **114** (1937), 570–590.

[17] C-Q Zhang, Cycle cover theorems and their applications, *in* "Graph Structure Theory," (N. Robertson and P. Seymour, Eds.), *Contemporary Mathematics* **147** (1993), 405–418.

[18] C-Q Zhang, Cycle covers and cycle decompositions of graphs, *Annals Discrete Math.* **55** (1993), 183–190.

[19] C-Q Zhang, On even circuit decompositions of eulerian graphs, *J. Graph Theory* **18** (1994), 51–57.

[20] C-Q Zhang, Hamiltonian weights and unique 3-edge-colorings of cubic graphs, *Journal of Graph Theory*, (to appear).