RESEARCH PAPER

Boundary search and simplex decomposition method for MDO problems with a convex or star-like state parameter region

Dapeng Wang · Greg F. Naterer · G. Gary Wang

Abstract One major challenge in multidisciplinary design optimization (MDO) is the presence of couplings among state parameters, which demands an iterative and often expensive system analysis (SA) process for each function evaluation in optimization. This paper offers a new perspective and proposes a corresponding method for solving MDO problems. The proposed method, named the boundary search and simplex decomposition method (BSSDM), geometrically captures the relation among coupled state parameters with a feasible state parameter region. Given the feasible state parameter region, the SA can be avoided during the optimization of the system objective function. To identify the feasible state parameter region, a search strategy is developed to find boundary points of the region. In the boundary search process, a collaboration model (CM) is applied to maintain the feasibility of samples with respect to the SA. In search of the system optimum in the feasible region, a robust simplex decomposition algorithm is developed for convex and star-like feasible state parameter regions. The BSSDM is tested with two numerical cases, one of which is an MDO problem constrained by a convex state parameter region, and the other is a SA problem with a star-like state parameter region. All results are then validated, and the results show the promising capability of the proposed BSSDM.

Keywords Multidisciplinary design optimization · Collaboration model · Convex/nonconvex state parameter region · Boundary search · Simplex decomposition

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1 Introduction

Multidisciplinary design optimization (MDO), which is also called multidisciplinary design synthesis, is an important design methodology in industry today. A general MDO problem is defined as

$$\min_{\boldsymbol{x}_s, \boldsymbol{x}} f(\boldsymbol{x}_s, \boldsymbol{x}_{cs}, \boldsymbol{y})$$

$$\text{subject to} : y_i = Y_i(\boldsymbol{x}_i, \boldsymbol{x}_{csi}, \boldsymbol{y}_{ci}), \ i = 1, \dots, n \qquad (1)$$

$$\boldsymbol{g}(\boldsymbol{x}, \boldsymbol{y}) \le 0$$

where

y_i a state parameter/variable given by its corresponding discipline

Y_i function associated with y_i

n number of state parameters

$\boldsymbol{y}$ $\{y_1, ..., y_i, ..., y_n\}$, a vector of state parameters

$\boldsymbol{y}_{ci}$ $\{y_j\}$, j≠i, a vector of state parameters output from other discplines to discpline/subsystem i

f system objective function

$\boldsymbol{x}_i$ a vector of disciplinary design variables of the state parameter y_i, and $\boldsymbol{x}_i \cap \boldsymbol{x}_j$, $i \neq j$ does not have to be $\varnothing$

$\boldsymbol{x}_{csi}$ a vector of system–subsystem design variables shared by both y_i and the objective function f

$\boldsymbol{x}_{cs}$ $\{\boldsymbol{x}_{cs1}, ..., \boldsymbol{x}_{csi}, ..., \boldsymbol{x}_{csn}\}$, a union of $\boldsymbol{x}_{csi}$, i=1, ..., n, and $\boldsymbol{x}_i \cap \boldsymbol{x}_{cs} = \varnothing$

$\boldsymbol{x}$ $\{\boldsymbol{x}_1, ..., \boldsymbol{x}_i, ..., \boldsymbol{x}_n, \boldsymbol{x}_{cs}\}$, a union of the disciplinary variables and system–subsystem design variables

$\boldsymbol{x}_s$ a vector of system design variables of f, $\boldsymbol{x}_s \cap \boldsymbol{x} = \varnothing$

$\boldsymbol{g}$ a vector of inequality constraints

In (1), $\boldsymbol{y}$ is governed by

$$\begin{cases} y_1 = Y_1(\boldsymbol{x}_1, \boldsymbol{x}_{cs1}, \boldsymbol{y}_{c1}) \\ \dots \\ y_i = Y_i(\boldsymbol{x}_i, \boldsymbol{x}_{csi}, \boldsymbol{y}_{ci}) \\ \dots \\ y_n = Y_n(\boldsymbol{x}_n, \boldsymbol{x}_{csn}, \boldsymbol{y}_{cn}) \end{cases} . \qquad (2)$$

A system analysis (SA) or the multidisciplinary analysis (MDA) is described in (2). The solution of (2) is usually calculated by an iterative procedure, such as the Gauss–Seidel iterative method for linear equations and the steepest descent method for nonlinear equations (see, e.g., Burden and Faires 2000). State parameters $\boldsymbol{y}$ usually represent some physical features in a physical discipline, and they are usually calculated by computationally intensive processes.

The standard MDO formulation in (1) is also called the multidisciplinary feasible (MDF) formulation in the nonlinear programming community, or the "all-in-one" formulation in the engineering community (Alexandrov and Lewis 2000c). It can be solved by conventional optimization algorithms, such as gradient-based methods, treating the SA/MDA as equality constraints. The main difficulty of applying the "all-in-one" formulation in practice is that the computational cost could be prohibitive because the SA/MDA is frequently called during the optimization process. Global sensitivity equations (GSE) were used to linearize nonlinear couplings in the neighborhood of a feasible design in the concurrent subspace optimization (CSSO) formulation by Sobieszczanski-Sobieski (1988, 1990), Renaud and Gabriele (1991, 1993), and Sellar and Batill (1996), and in the Bi-level integrated system synthesis (BLISS) formulation by Sobieszczanski-Sobieski et al. (2000). Based on the similar idea of using slack variables in the "all-at-once" formulation (Alexandrov and Lewis 2000c), Braun and Kroo (1997) developed another decomposition-based MDO formulation named collaborative optimization (CO) by using slack variables for decoupling. In CO, the interdisciplinary consistency is held in the system level by extra equality

constraints, which are the match between the slack variables issued from the system level and their corresponding variables evaluated and returned from disciplinary analyses. Alexandrov and Lewis (1999, 2000a, b) found that CO has difficulties preserving the standard Karush–Kuhn–Tucker condition due to its formulation structure.

The main difficulties of solving MDO problems are primarily raised by the couplings among state parameters from different disciplines. When a discipline involves a computationally expensive process, the computation of MDO becomes cost-prohibitive. Reducing the turn-around time of design and achieving the global optimal design are the main goals of all MDO methodologies. This paper proposes a new MDO method, named the boundary search and simplex decomposition method (BSSDM). The BSSDM can geometrically capture the relation among coupled state parameters by exploring the feasible region formed by state parameters, in which any point (a set of state parameters) will satisfy (2). Given the captured coupling information, a robust optimum solution of MDO problems can be achieved easily and efficiently.

In the following sections, the motivation of the BSSDM will be elaborated first in Section 2. Then, a collaboration model (CM) developed by Wang (2005a) for checking the samples' feasibility with respect to the SA/MDA will be reviewed in Section 3. The infrastructure of the BSSDM will be introduced in Section 4. Finally, the results of test cases solved with the BSSDM will be shown and discussed in Section 5.

2 Motivation

The BSSDM is developed to address dominant couplings among state parameters in MDO problems. The relation among coupled state parameters is not a one-to-one mapping. Given the values of $\boldsymbol{y}_{c1}$ in (2), the value of y_1 can vary within a certain range due to many feasible combinations of $\boldsymbol{x}_1$ and $\boldsymbol{x}_{cs1}$. This fact indicates that one state parameter cannot be approximated with a simple fitting function of its own variables. In other words, such a fitting function only captures part of the relation among coupled state parameters. The relation among n-coupled state parameters can be described by an nD feasible state parameter region. Geometrically, the feasible state parameter region can be further classified into convex and nonconvex types. Correspondingly, the SA/MDA can be classified into convex and nonconvex problems.

It would be beneficial to know the feasible boundary of state parameters when solving MDO problems. In the BSSDM, the feasible state parameter region will be approximated based on boundary points, which are itera-

tively explored by a boundary search strategy through a random sampling process. The approximated feasible state parameter region can be further decomposed into simplexes to show the relation among coupled state parameters explicitly and geometrically. The BSSDM gives a novel perspective of describing the interdisciplinary couplings and obtaining the solution of an MDO problem.

The BSSDM focuses on certain MDO problems, in which the system objective function, f, is only a function of state parameters, y, and system design variables, x_s. The MDO problem formulation required by the BSSDM is defined by

$$\min_{x_s,x} f(x_s, y)$$
$$\text{subject to}: y_i = Y_i(x_i, y_{ci}), \ i = 1, \ldots, n \quad , \quad (3)$$
$$g_c(x, y) \leq 0$$
$$g_s(x_s, y) \leq 0$$

where g_c is a vector of disciplinary constraints, g_s is a vector of system constraints, and x is a union of x_i, i=1, ..., n. The only difference between (3) and (1) is that the MDO formulation defined in (3) does not involve the system–subsystem variables, x_{cs}. It is, thus, a limitation of the BSSDM in terms of the applicability to MDO problems. When the relation among coupled state parameters is explicitly available, optimizing the system objective function does not need to call disciplinary analyses based on the formulation defined in (3). In (3), having known the feasible region of coupled state parameters, the SA/MDA can be replaced by simple uncoupled constraints. Therefore, the burden of calling the SA/MDA can be eliminated for optimizing f. Also, the problem defined by (3) is assumed to have a convex or star-like state parameter region at the current development state of the BSSDM. In this paper, the term, 'convex region,' is equivalent to the term 'convex set' in mathematics.

Fundamentally, the BSSDM is a sampling-based method. The feasible state parameter region is explored and approximated by some selected samples from a sample pool. In a sampling process, it is not certain if a sample can produce feasible values of its coupled state parameters governed by (2). A collaboration model (CM) is applied for choosing feasible samples and will be reviewed in the next section.

3 Review of CM

Wang (2005a) proposed a CM for checking the feasibility of samples with respect to the SA/MDA. CM can output a distribution of samples that differentiates samples in terms of the possibility being feasible with respect to the SA/MDA. It

can be viewed as a filter for a sampling process. The CM is illustrated by a two-state-parameter SA/MDA defined in (4) according to the formation of MDO problems in (3).

$$y_1 = Y_1(x_1, y_2)$$
$$y_2 = Y_2(x_2, y_1) \quad . \quad (4)$$

In (4), y_1 is an explicit function of x_1 and y_2. The explicit expression reflects the physical relation between y_1 and its variables, i.e., x_1 and y_2. Meanwhile, y_1 is implicitly affected by x_2, which are the design variables from the other discipline. The situation is similar for y_2. Therefore, mathematically, all state parameters, y, are affected by x, the union of all associated design variables x_1 and x_2. The mathematical function of a state parameter y_i with respect to x uncovers the intrinsic mathematical dependency among coupled disciplines, and it plays an important role in the CM. A linear radial basis function (RBF) shown below is used to implement the CM:

$$\widetilde{F}(X) = \sum_{e=1}^{E} \alpha_i \|X - X^{(e)}\| \quad , \quad (5)$$

where $\widetilde{F}$ is the approximation of F, such as $\bar{y}_i$ and $\bar{\bar{y}}_i$ defined in (6) and (7); X is a design variable vector, such as $\widetilde{x}_i$ in (6), and $\bar{y}_{ci}$ and x_i in (7); $\{\varphi(\|X - X^{(e)}\|), \ e = 1, 2, \ldots, E\}$ is a set of E functions known as RBFs; $\|\cdot\|$ denotes a norm, which is usually Euclidean; and α_i are unknown coefficients (weights) calculated by a set of simultaneous linear equations, $\widetilde{F}(X^{(e)}) = F^e$. The reason to choose the RBF is because the RBF approximation goes through each experimental point and, therefore, gives better accuracy in the neighborhood of experimental points. In general, other approximation techniques can also be used, such as Kringing modeling.

The proposed CM is built based on two dependent approximations of coupled state parameters with the RBF in (5) as

$$\bar{y}_i = \overline{Y}_i(\widetilde{x}_i), \ i = 1, \ldots, n \quad (6)$$

$$\bar{\bar{y}}_i = \overline{\overline{Y}}_i(x_i, \bar{y}_{ci}), \ i = 1, \ldots, n \quad , \quad (7)$$

where $\widetilde{x}_i$ includes all the associated design variables of the state parameter y_i. The value of the approximate function of y_i is represented by $\bar{y}_i$ in (6). The approximate value of y_i marked as $\bar{\bar{y}}_i$ can be explicitly evaluated by (7), given x_i and $\bar{y}_{ci}$. Provided that the approximation is sufficiently accurate, and a set of design variables, x, simultaneously satisfies (6) and (7), the difference between $\bar{y}_i$ and $\bar{\bar{y}}_i$ ought to be small; otherwise, the difference will be large.

For a given set of design variables, x, the interdisciplinary consistency/discrepancy of state parameters, D, is defined by

$$D = \sum_{i=1}^{n} \left| \bar{y}_i - \bar{\bar{y}}_i \right| . \tag{8}$$

If the value of D is small, this set of design variables, x, is likely feasible with respect to the SA/MDA.

The CM defined in (6), (7), and (8) gives a distribution of the interdisciplinary discrepancy/consistency to a group of samples with respect to the SA/MDA. It means that samples with a smaller value of D are more likely feasible with respect to the SA/MDA than those with a larger value of D. The CM was also applied in a collaboration pursuing method (CPM) for solving a general MDO problem defined in (1) (by Wang et al. 2005). The effectiveness of the CM is demonstrated by applying the CM to a problem, which is the SA/MDA of test case 1 (in Section 5) defined by

$$y_1 = x_1 + x_2 - 2 + (y_2/1.5)^4$$
$$y_2 = x_3 + x_4 - 2 + (y_1/1.8)^4 \tag{9}$$
$$\text{subject to}: 1 \le x_1, x_2, x_3, x_4 \le 1.9$$

One hundred samples are generated through a random sampling process in the design variable space, (x_1, x_2, x_3, x_4), for studying the effectiveness of the CM. Five experimental points listed in Table 1 are randomly generated and used for RBF approximations defined in (6) and (7). After applying the CM for calculating the interdisciplinary discrepancy/consistency value, D, of the 100 random samples, the distribution of D over the 100 samples is depicted in an ascending order in terms of the value of D in Fig. 1. The feasibility of the 100 samples with respect to the SA/MDA in Fig. 1 was determined based on the evaluations of y of the 100 samples by calling the SA/MDA. Samples marked with '•' are feasible, and samples plotted with '+' signs are infeasible. As expected, the samples with a small value of D in Fig. 1 are more likely feasible with respect to the SA/MDA than other samples with a large value of D. The same study was also applied to cases with 500 and 1,000 random samples, as shown in Figs. 2 and 3.

The effectiveness of the CM was also intensively studied by Wang (2005b). It was shown that the prediction of feasible samples with respect to the SA/MDA based on the D distribution given by the CM becomes more accurate with more experimental points. In the next section, the infrastructure of the BSSDM will be elaborated in detail.

4 Boundary search and simplex decomposition method

As shown in Fig. 4, the BSSDM consists of five modules, which are the boundary search process (BSP), domain decomposition, boundary refining, system optimization, and disciplinary analysis. The BSP module involves iterative processes to search for boundary points. The domain decomposition module approximates the actural feasible state parameter region based on the boundary points from the BSP and divides the approximated region into simplexes. The objective function f is optimized by the system optimization module within each simplex individually, and the optimum solutions of x_s and y can be achieved by comparing solutions from each simplex. The boundary refining module provides options to further improve the approximation of the actual state parameter region. Eventually, the optimum value of x is evaluated by a trace back process in the disciplinary analysis module, given the optimum solutions of x_s and y from the system optimization module. Before introducing the flow chart of the proposed method as illustrated in Fig. 4, some key techniques are first discussed, i.e., the boundary search strategy and domain decomposition method.

4.1 Boundary search strategy

The boundary search strategy is applied after a limited number of initial feasible points are identified. It explores the boundary points of the actual feasible state parameter region based on which the state parameter region can be approximated with simplexes. How the initial feasible points are obtained will be described later in Section 4.3. Three boundary search strategies, which are the stair-climbing strategy, strip-measure strategy, and explosion strategy, were developed and discussed by Wang (2005a). In the BSSDM, the explosion strategy is applied due to its efficiency, which will be explained in this section.

Table 1 Experimental points for the RBF approximation

| Experiment	x_1	x_2	x_3	x_4	y_1	y_2
1	1.306581	1.398711	1.007338	1.048180	0.705300	0.079091
2	1.430430	1.504102	1.106891	1.172904	0.937613	0.353416
3	1.472354	1.571369	1.160585	1.242747	1.058492	0.522912
4	1.047014	1.668556	1.249554	1.318132	0.740550	0.596337
5	1.108217	1.734028	1.302309	1.383587	0.904644	0.749696

Fig. 1 Distribution of D—interdisciplinary consistency/discrepancy of 100 random samples
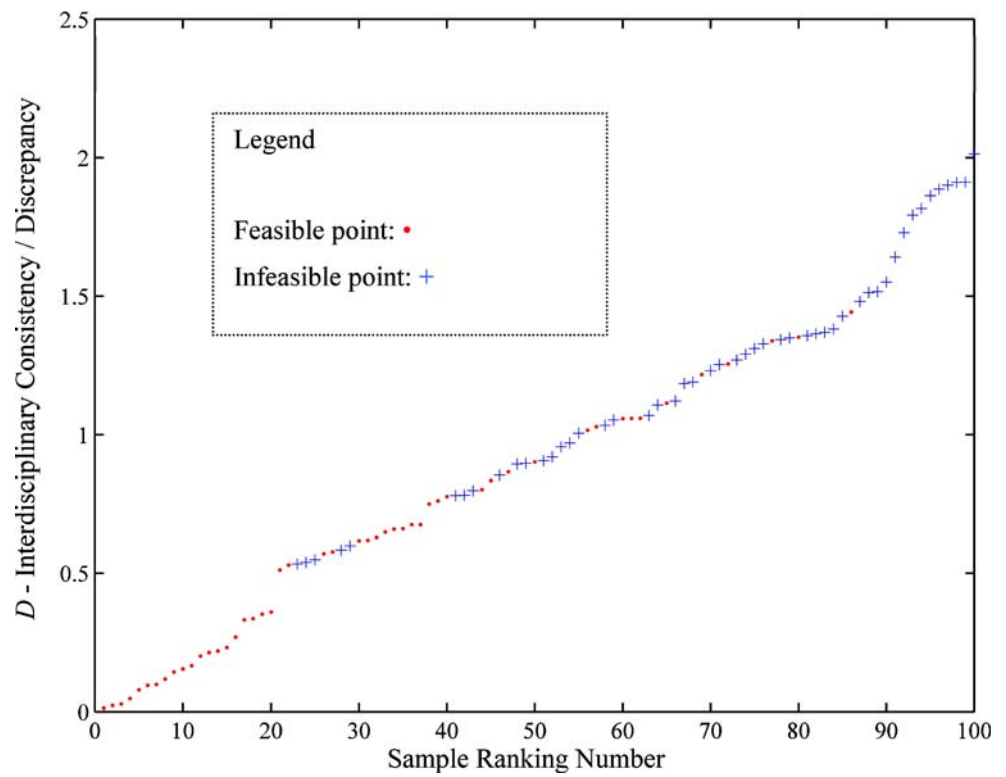
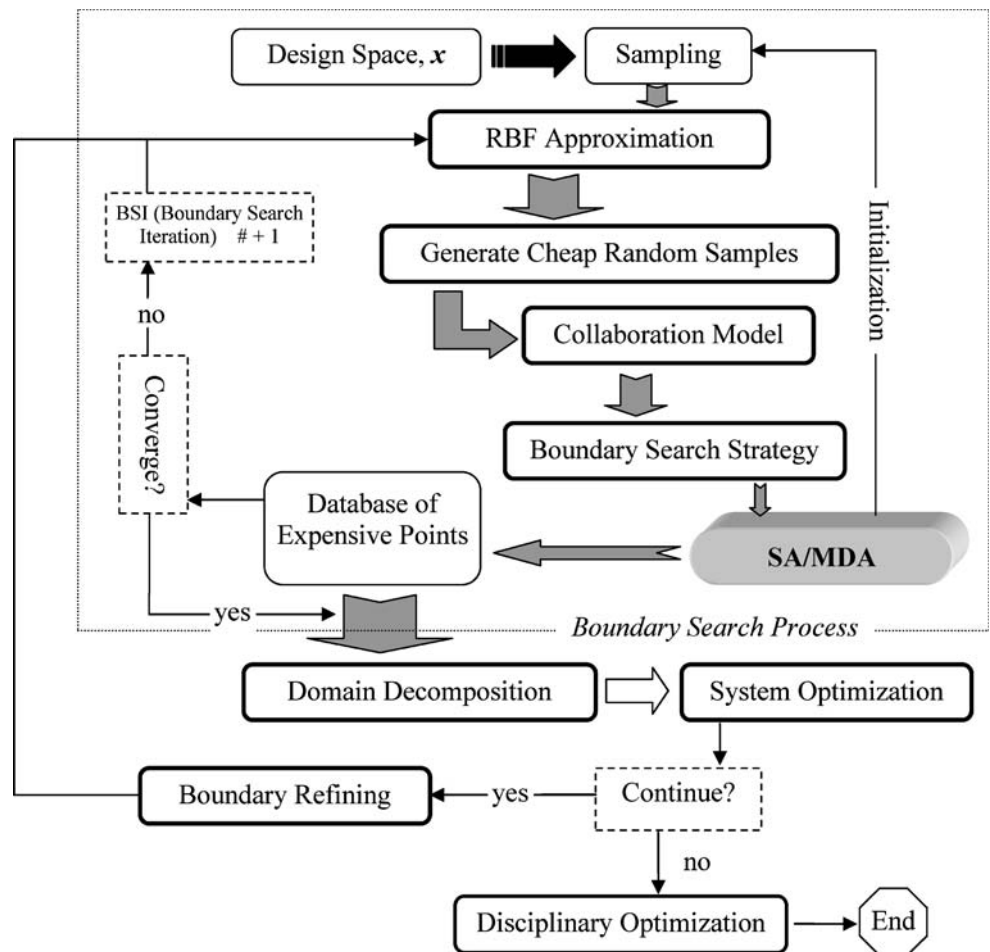**Fig. 2** Distribution of D—interdisciplinary consistency/discrepancy of 500 random samples
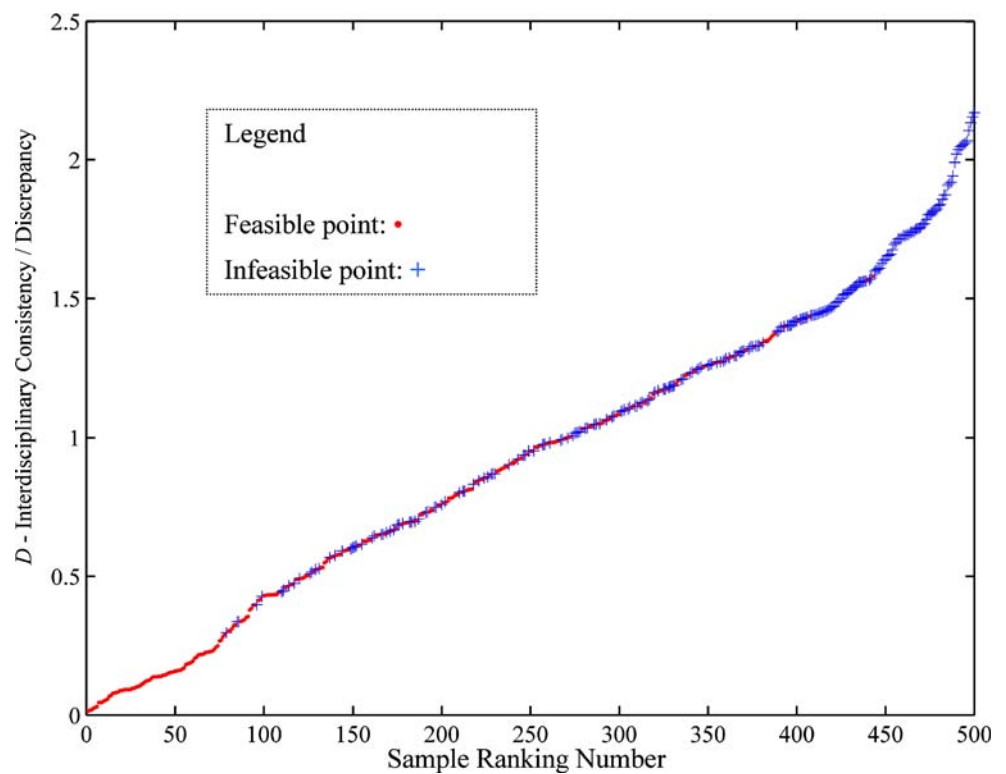
Fig. 3 Distribution of D—inter-disciplinary consistency/discrepancy of 1,000 random samples
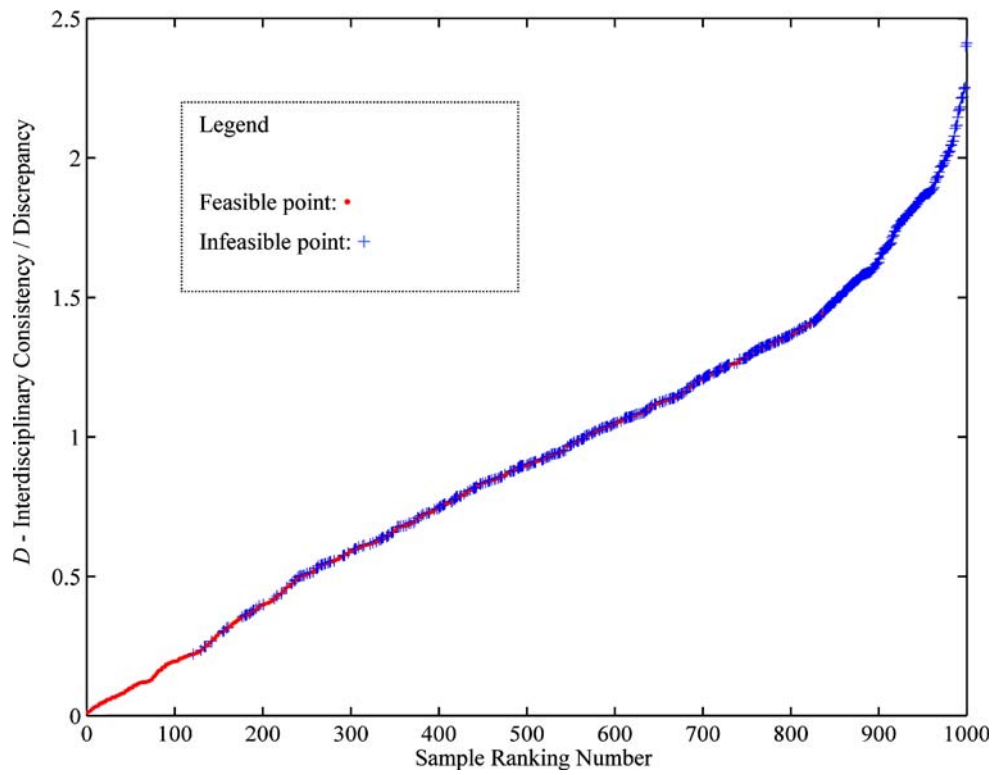
Fig. 4 Architecture of the boundary search and simplex decomposition method

Based on (3), y is constrained by

$$\begin{cases} y_1 = Y_1(\boldsymbol{x}_1, \boldsymbol{y}_{c1}) \\ \ldots \\ y_i = Y_i(\boldsymbol{x}_i, \boldsymbol{y}_{ci}) \\ \ldots \\ y_n = Y_n(\boldsymbol{x}_n, \boldsymbol{y}_{cn}) \end{cases} . \tag{10}$$

Literally, the explosion strategy is analogous to an explosion process. The search procedure starts with some initial feasible design points obtained in the state parameter space and explores the furthest point with respect to a geometric center along different directions. Given K feasible state parameter points in the state parameter space, e.g., points marked with 'filled squares' in a two-state parameter case in Fig. 5, the geometric center ce is defined by

$$y_{ce,i} = \frac{\max\left(y_{k,i}, \ k = 1, \ldots, \ K\right) + \min\left(y_{k,i}, \ k = 1, \ldots, \ K\right)}{2},$$

$$i = 1, \ldots, n , \tag{11}$$

First, the space of state parameters can be divided into quadrants with respect to the geometric center ce. For example, four quadrants in a 2D space can be defined by

$$\begin{array}{l} \text{Quadrant 1:} \ y_1 \geq y_{ce,1} \text{ and } y_2 \geq y_{ce,2} \\ \text{Quadrant 2:} \ y_1 \leq y_{ce,1} \text{ and } y_2 \geq y_{ce,2} \\ \text{Quadrant 3:} \ y_1 \leq y_{ce,1} \text{ and } y_2 \leq y_{ce,2} \\ \text{Quadrant 4:} \ y_1 \geq y_{ce,1} \text{ and } y_2 \leq y_{ce,2} \end{array} . \tag{12}$$

The number of quadrants, Q, is equal to 2^n. Then, in each quadrant, the feasible boundary is pursued in (13) by searching for a sample point (from a large sample pool) with the longest Euclidean distance with respect to the geometric center ce.

$$\max\left(L_1, \ldots, L_s, \ldots, L_{p_q}\right) , \tag{13}$$

where P_q is the number of samples in quadrant q. L_s denotes the Euclidean distance between sample s and ce and is defined by

$$L_s = \sqrt{\sum_{i=1}^{n} \left(\bar{y}_{s,i} - y_{ce,i}\right)^2} , \tag{14}$$

where $\bar{y}_{s,i}$ is the approximate value of y_i calculated by (6). Meanwhile, the center ce will be dynamically refreshed in the next boundary search iteration (BSI) with more feasible design points. Finally, the furthest vertex in each quadrant is found, e.g., points marked with 'filled circles' in Fig. 5.
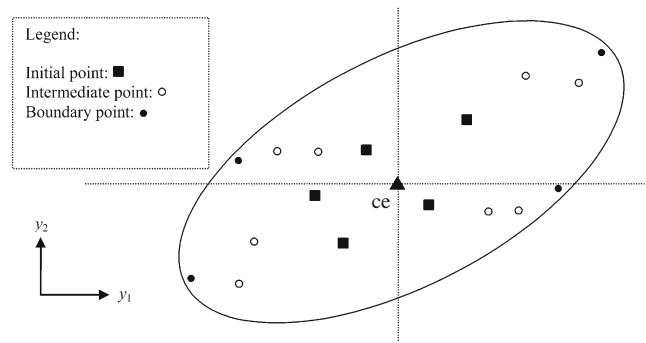

Fig. 5 Explosion strategy applied in a 2D state parameter region

All available feasible points are considered to be boundary point candidates and will be connected by facets to build up a convex hull in the domain decomposition module, which is explained in the next section.

4.2 Domain decomposition

Suppose that (10) has a feasible convex state parameter region. Given boundary points of the feasible convex state parameter region, the boundary of the feasible region can be tessellated with facets. Each facet consists of n vertices. In the 2D state parameter space [n=2 in (10)], a facet is a straight line segment, while in a 3D space, it is a triangle. A facet and an interior point in the feasible region construct a simplex, which is, for example, a triangle in the 2D space and a tetrahedron in the 3D space. In this way, a convex region can be uniquely decomposed into simplexes centered about a common interior point in the region. Geometrically, each simplex is confined by its facets. In an nD space, a facet can be described by

$$a_1 y_1 + \ldots, + a_n y_n = a_{n+1} , \tag{15}$$

where a_i (i=1, ..., n+1) can be uniquely determined by n points. For example, in Fig. 6, the facet AB can be defined by state parameter points A and B. For constructing a
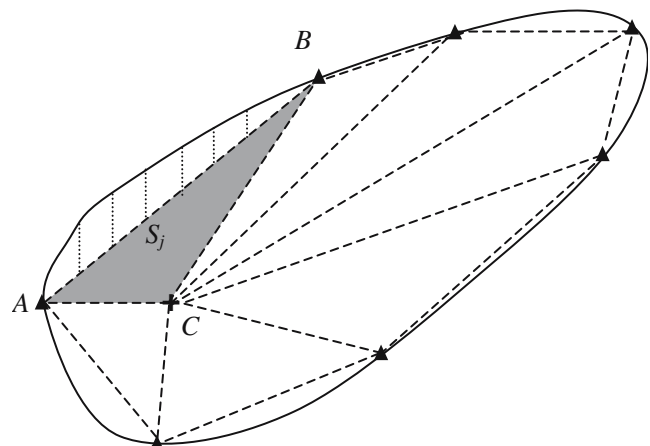

Fig. 6 Simplex convex decomposition method for a 2D state parameter case

simplex, the equal sign in (15) will be changed to either "≤" or "≥" by substituting a vertex of the simplex into a facet (of the same simplex), which does not pass through the vertex, e.g., substitution of point C into facet AB in Fig. 6. Eventually, in the nD state parameter space, a simplex can be defined with (n+1) linear constraints by

$$a_{1,1}y_1 + a_{2,1}y_2 + \ldots + a_{n,1}y_n \leq \text{or} \geq a_{n+1,1}$$
$$\ldots \qquad\qquad\qquad\qquad\qquad\qquad\qquad .$$
$$a_{1,n+1}y_1 + a_{2,n+1}y_2 + \ldots + a_{n,n+1}y_n \leq \text{or} \geq a_{n+1,n+1}$$
$$(16)$$

A Quickhull algorithm is used to find boundary points from all feasible state parameter points prepared by the explosion strategy and to connect all boundary points to generate facets enclosing the smallest convex hull that contains all feasible state parameter points (Barber et al. 1996). This process was implemented using a function named *convhulln* in MATLAB® 6.0 (MathWorks, Natick, MA, USA). The *convhulln* function returns the indices of the boundary points in a data set that comprises of the facets of the convex hull for the set.

Due to its complexity, this paper has no intention to handle a general nonconvex region. For certain situations, namely star-like regions, a star-like decomposition algorithm is proposed and studied in test case 2 in Section 5. A set S is star-like with respect to some $\widehat{x} \in S$ iff the segment from $\widehat{x}$ to any other point in S is contained in S. A star-like region case is shown in Fig. 7. The pseudo-code of the proposed star-like decomposition algorithm is elaborated as follows by using Fig. 7 as an example:

1. Given all current available feasible points, e.g., points A, B, C, D, E, and F, apply the Quickhull algorithm to construct a convex hull and decompose the region into simplexes, such as S_1.
2. Find a point (representing the furthest vertex in a quadrant) not listed on any facet given by step 1, e.g., C. Locate this point in one of the current simplexes
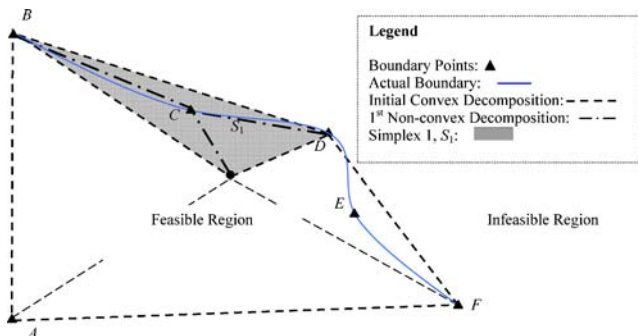

Fig. 7 Illustration of the star-like decomposition algorithm for a 2D state parameter case

based on (16). For example, substitute this point into (16). If (16) is satisfied, this point must be located in the simplex, i.e., S_1. Break up this simplex and reconstruct it by new facets made of any different $n-1$ vertices of S_1 (excluding the interior vertex), i.e., B or D, and this boundary point, i.e., C. Update all simplexes and repeat step 2 until all vertices in quadrants are listed on one of the facets.

With the understanding of the boundary search strategy and boundary decomposition, the flow chart of the BSSDM will be explained next.

4.3 Procedure of the BSSDM

The BSSDM starts with the BSP module to search for boundary points. As shown in Fig. 4, an initialization procedure first creates some initial experimental points through a random sampling process. These experimental points are obtained by calling the expensive SA/MDA, and therefore, they are referred to as expensive points. Using these expensive points, two different RBF models, $\bar{y}_i$ and $\bar{\bar{y}}_i$, are then constructed for each discipline based on the CM defined in (6) and (7). Then a larger number of sample points, e.g., p=10^4, are randomly generated in the design variable space, $[x_{Lb}, x_{Ub}]$. The state parameters of the p random samples are approximated with the RBF models. These random points with approximated state parameters are referred to as cheap points. Therefore, we have

$$\bar{y}_{q,i} = \overline{Y}_i(\widetilde{x}_{q,i}), \quad i = 1, \ldots, n, \quad q = 1, \ldots, p \qquad (17)$$

$$\bar{\bar{y}}_{q,i} = \overline{\overline{Y}}_i(x_{q,i}, \bar{y}_{q,ci}), \quad i = 1, \ldots, n, \quad q = 1, \ldots, p. \qquad (18)$$

The interdisciplinary discrepancy of state parameters for each sample point is evaluated by

$$D_q = \sum_{i=1}^{n} |\bar{y}_{q,i} - \bar{\bar{y}}_{q,i}|, \quad q = 1, \ldots, p. \qquad (19)$$

Based on the RBF approximation models, the constraint check (for each sample) is implemented by

$$\bar{g}_{c,q}(x_q, \bar{y}_q) \leq 0, \quad q = 1, \ldots, p \qquad (20)$$

$$\bar{\bar{g}}_{c,q}(x_q, \bar{\bar{y}}_q) \leq 0, \quad q = 1, \ldots, p \qquad (21)$$

Samples violating constraints defined in (20) and (21) are eliminated from the sample set. Then, the discrepancy check takes place to filter out infeasible samples with

respect to SA/MDA according to a pre-set threshold[1] of the interdisciplinary discrepancy D, e.g., 1,000. The remaining samples, e.g., p_1 samples, will be allocated into each quadrant according to (11) and (12). Finally, a sample with the largest Euclidean distance as defined in (13) and (14) in each quadrant is chosen to be evaluated expensively by calling the SA/MDA, and a BSI is ended. True feasible points will then be used as new expensive points, which are combined with existing expensive points to construct more accurate approximation models in the next BSI, as shown in Fig. 4. The BSP stops if the position of the geometric center of the explored state parameter region stays stationary according to a movement criterion over a certain number of consecutive BSIs, e.g., two. The movement criterion of ce is defined by

$$\sqrt{\sum_{i=1}^{n} \left(y_{ce,i}^{(t)} - y_{ce,i}^{(t+1)} \right)^2} \leq 1e^{-3} \tag{22}$$

or

$$\left| ce^{(t)} - ce^{(t+1)} \right| \leq 1e^{-3}, \tag{23}$$

where t indicates the tth BSI.

Once the BSP is terminated, one would have a collection of expensive points including boundary points. Then, a convex hull can be constructed by applying the Quickhull method in the domain decomposition module. Also a nonconvex region check follows. As a result, geometrically, the feasible state parameter region is decomposed into simplexes confined by linear constraints as defined in (16).

As points in the feasible state parameter region will satisfy (10) and the equality constraints in (3) can be replaced by simplexes defined by (16), the optimization of the system objective function f can then be performed within each simplex with respect to y and x_s in the system optimization module. The system optimization, thus, becomes

$$\min_{x_s,y} f(x_s, y)$$

$$\text{subject to}: a_{1,1}y_1 + a_{2,1}y_2 + \ldots + a_{n,1}y_n \leq \text{ or } \geq a_{n+1,1}$$

$$\ldots$$

$$a_{1,n+1}y_1 + a_{2,n+1}y_2 + \ldots + a_{n,n+1}y_n \leq \text{ or } \geq a_{n+1,n+1}$$

$$g_s = G_s(x_s, y) \leq 0 \tag{24}$$

As a result, the system optimization module outputs optimum values: y^*, f^*, and x_s^*. It is to be noted that the expensive SA/MDA process is not called at all when optimizing (24).

[1] The value of the threshold depends on the problem. In general, a small threshold results in a conservative filtering process; a large one could take some infeasible samples. The threshold value can be determined based on users' judgment by referring to the D distribution given by (19).

The best value of the system objective can be further improved in the boundary refining module by continually exploring undiscovered regions between current facets and the actual boundary. In the BSSDM, the undiscovered region between a facet (belonging to the simplex containing the current best value of f) and its corresponding actual boundary segment, e.g., the cross hatched area in Fig. 6, is chosen for further investigation. This is because the actual optimum solution might be in a small neighborhood of the current best solution in the state parameter space for continuum problems. Ideally, all undiscovered regions could be further explored if the cost is of little concern. The boundary refining module geometrically defines the undiscovered regions by switching the sign of facets defined in (16), e.g., facet AB in Fig. 6. With defined undiscovered regions, the actual boundary of the undiscovered regions will be explored over a number of BSIs in a new BSP. The convergence criterion of BSSDM is the area difference between approximated regions from two adjacent BSPs. It means that the boundary refining process stops if the new discovered area is small enough subject to time and budget.

Given y^* and f^*, the optimum value of disciplinary design variables, x^*, can be traced back by running an optimization process in the disciplinary optimization module as follows:

$$\min_x S = \sum_{i=1}^{n} \left(y_i^* - y_i \right)^2$$

$$\text{subject to}: y_i = Y_i(x_i, y_{ci}^*), \ i = 1, \ldots, n \tag{25}$$

$$g_c(x, y^*) \leq 0$$

When $x_i \cap x_j \neq \varnothing (i \neq j)$, the computational expense of solving (25) is close to calling the SA/MDA once. In the case where $x_i \cap x_j = \varnothing (i \neq j)$, x_i can be evaluated independently with an optimization process in each subsystem as follows:

$$\min_{x_i} \left(y_i^* - y_i \right)^2$$

$$\text{subject to}: y_i = Y_i(x_i, y_{ci}^*), \tag{26}$$

$$g_{c,i}(x_i, y^*) \leq 0$$

where $g_{c,i}$ is a sub-vector of g_c. Eventually, the optimum solution is given as y^*, x^*, x_s^*, and f^*. A detailed flow chart of the BSSDM is shown in Fig. 8. Two test cases are solved successfully with the BSSDM, and test results are shown and discussed in the next section.

5 Test problems and discussion

The BSSDM is tested with two cases. The authors formulated the first numerical test case by placing two coupled state parameters into the well-known six-hump camelback prob-

Fig. 8 Flow chart of the boundary search and simplex decomposition method
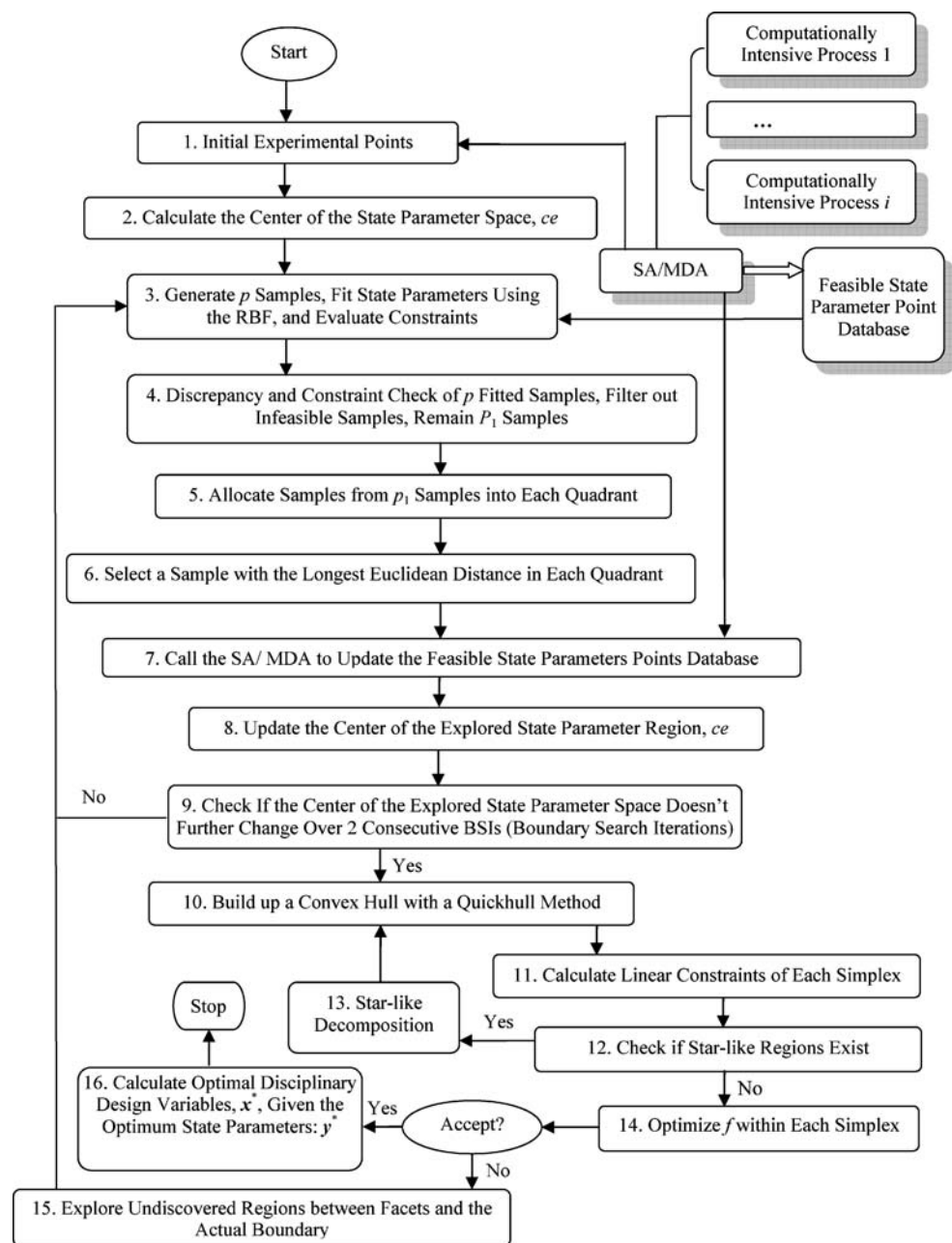
lem. Two BSPs were implemented in test case 1. Test case 2 is a problem of the SA/MDA in Tappeta et al. (1998) and is applied to demonstrate the ability of the BSSDM in dealing with a star-like state parameter region. One BSP is implemented in this problem. Both cases are validated by enumerations. The 'fsolve' function in MATLAB® 6.0 (MathWorks) was applied to solve the SA/MDA with its default convergence tolerance value, 10^{-4}, and the 'fmincon' function in MATLAB® 6.0 (MathWorks) is used for optimization in disciplinary optimization module. Based on the authors' experience of using BSSDM and CPM, 10^4 random samples are usually enough for solving MDO problems with the number of design variables up to ten. In step 3, in Fig. 8, p is set to 10^4 for both test cases.

5.1 Test case 1

Test case 1 is formulated as follows:

$$\min f = 4y_1^2 - 2.1y_1^4 + \frac{y_1^6}{3} + y_1 y_2 - 4y_2^2 + 4y_2^4$$

$$\text{subject to}: y_1 = x_1 + x_2 - 2 + \left(y_2/1.5\right)^4$$

$$y_2 = x_3 + x_4 - 2 + \left(y_1/1.8\right)^4 \tag{27}$$

$$y_1 \leq 1.0$$

$$y_2 \leq 1.0$$

$$1 \leq x_1, x_2, x_3, x_4 \leq 1.9$$

In total, ten independent runs are carried out for test case 1 to examine the robustness of the BSSDM. For

each run, the BSSDM started with five initial expensive points that are feasible, with respect to both the SA/MDA and the other constraints. All runs were terminated after two BSPs. All runs' first BPS is converged when there is no further movement of the geometric center ce after two consecutive BSIs. The second BSPs of runs 1–5, 6–7, and 8–10 are converged after 2, 3, and 4 consecutive BSIs, respectively, without further movement of the geometric center ce. The results of test case 1 given by the BSSDM are shown in Table 2. In Table 2, each BSP output one best experimental point; then, f was optimized within simplexes after each BSP, and the optimal solutions of x were evaluated by (25), given y^* after two BSPs. For run

1, the complete process information of solving test case 1 is depicted in Fig. 9, which shows all experimental points for exploring the actual boundary, the actual state parameter region given by enumeration, the boundary of the feasible state parameter region approximated by the BSSDM, and the contour of f. Also, the cumulative number of the SA/MDA and the convergence history of each BSP are shown in Figs. 10 and 11, respectively. The optimal value given by the BSSDM is very close to the actual optimum according to the contour of f. Besides the optimum solution of f, the BSSDM almost gives a full geometric description of the relation between coupled state parameters.

Table 2 Results of test case 1

| No. of run	No. of SA/MDA			Solutions					
	Feasible	Infeasible		1st boundary search process			2nd boundary search process		
		Initial	After initialization	Best experiment		Intermediate	Best experiment		Optimum solution
1	55	10	1	y_1=0.194433 y_2=0.934829 f=−0.110784 x_1=1.025010 x_2=1.018566 x_3=1.436627 x_4=1.498066		y_1=0.145198 y_2=0.674395 f=−0.810511	y_1=0.069245 y_2=0.678303 f=−0.927530 x_1=1.025179 x_2=1.002251 x_3=1.010931 x_4=1.667369		y_1=0.069245 y_2=0.678303 f=−0.927530 x_1=1.013715 x_2=1.013715 x_3=1.339150 x_4=1.339150
2	42	8	2	y_1=0.184069 y_2=0.918080 f=−0.227642 x_1=1.028258 x_2=1.015479 x_3=1.407857 x_4=1.510113		y_1=0.153910 y_2=0.681099 f=−0.796379	y_1=0.080887 y_2=0.739779 f=−0.905142 x_1=1.006350 x_2=1.015375 x_3=1.406123 x_4=1.333652		y_1=0.069443 y_2=0.685478 f=−0.929529 x_1=1.012915 x_2=1.012915 x_3=1.342738 x_4=1.342738
3	60	16	9	y_1=0.190143 y_2=0.699300 f=−0.724663 x_1=1.041600 x_2=1.101305 x_3=1.060560 x_4=1.638615		y_1=0.107519 y_2=0.686665 f=−0.876963	y_1=0.111801 y_2=0.747984 f=−0.852553 x_1=1.021980 x_2=1.027991 x_3=1.097539 x_4=1.650430		y_1=0.089601 y_2=0.681040 f=−0.901763 x_1=1.023553 x_2=1.023553 x_3=1.340517 x_4=1.340517
4	50	10	5	y_1=0.160730 y_2=0.921624 f=−0.261631 x_1=1.004504 x_2=1.013714 x_3=1.585420 x_4=1.336140		y_1=0.139306 y_2=0.687412 f=−0.824386	y_1=0.095167 y_2=0.673956 f=−0.891423 x_1=1.048804 x_2=1.005610 x_3=1.415693 x_4=1.258255		y_1=0.095530 y_2=0.675328 f=−0.891437 x_1=1.027222 x_2=1.027222 x_3=1.337660 x_4=1.337660
5	47	15	11	y_1=0.010465 y_2=0.217387 f=−0.177382 x_1=1.002001 x_2=1.008023 x_3=1.104855 x_4=1.112532		y_1=0.113781 y_2=0.675806 f=−0.864179	y_1=0.034693 y_2=0.438807 f=−0.601868 x_1=1.012382 x_2=1.014987 x_3=1.402943 x_4=1.035864		y_1=0.099327 y_2=0.673386 f=−0.885187 x_1=1.029356 x_2=1.029356 x_3=1.336688 x_4=1.336688
6	39	17	0	y_1=0.068446 y_2=0.249638 f=−0.197962 x_1=1.011269 x_2=1.056410 x_3=1.162730 x_4=1.086906		y_1=0.139213 y_2=0.677038 f=−0.822084	y_1=0.072176 y_2=0.648805 f=−0.907395 x_1=1.001643 x_2=1.035531 x_3=1.420962 x_4=1.227841		y_1=0.079895 y_2=0.670979 f=−0.911030 x_1=1.019928 x_2=1.019928 x_3=1.335487 x_4=1.335487
7	48	1	4	y_1=0.386583 y_2=0.723602 f=−0.166042 x_1=1.267575 x_2=1.064854 x_3=1.281192 x_4=1.440282		y_1=0.161793 y_2=0.659253 f=−0.772962	y_1=0.056208 y_2=0.703156 f=−0.947736 x_1=1.002180 x_2=1.005740 x_3=1.479164 x_4=1.223990		y_1=0.055455 y_2=0.695945 f=−0.948145 x_1=1.004558 x_2=1.004558 x_3=1.347972 x_4=1.347972
8	82	2	6	y_1=0.185760 y_2=0.425766 f=−0.379029 x_1=1.118326 x_2=1.060944 x_3=1.273765 x_4=1.151887		y_1=0.123827 y_2=0.677300 f=−0.848481	y_1=0.059706 y_2=0.705744 f=−0.943615 x_1=1.000716 x_2=1.009987 x_3=1.238280 x_4=1.467463		y_1=0.058771 y_2=0.696113 f=−0.944346 x_1=1.006194 x_2=1.006194 x_3=1.348056 x_4=1.348056
9	66	14	5	y_1=0.121406 y_2=0.589304 f=−0.776659 x_1=1.081105 x_2=1.016478 x_3=1.220964 x_4=1.368320		y_1=0.108950 y_2=0.682429 f=−0.873761	y_1=0.093389 y_2=0.766968 f=−0.862502 x_1=1.015831 x_2=1.009207 x_3=1.135758 x_4=1.631203		y_1=0.083716 y_2=0.690573 f=−0.912122 x_1=1.019396 x_2=1.019396 x_3=1.345284 x_4=1.345284
10	65	6	5	y_1=0.151832 y_2=0.745251 f=−0.783471 x_1= 1.022522 x_2=1.068378 x_3=1.335046 x_4=1.410154		y_1=0.118877 y_2=0.667192 f=−0.852541	y_1=0.086392 y_2=0.654171 f=−0.892977 x_1=1.024698 x_2=1.025519 x_3=1.480990 x_4=1.173176		y_1=0.065365 y_2=0.671718 f= −0.929517 x_1=1.012575 x_2= 1.012575 x_3=1.335858 x_4=1.335858

Fig. 9 Complete information of the results of run no. 1 in test case 1 given by the BSSDM
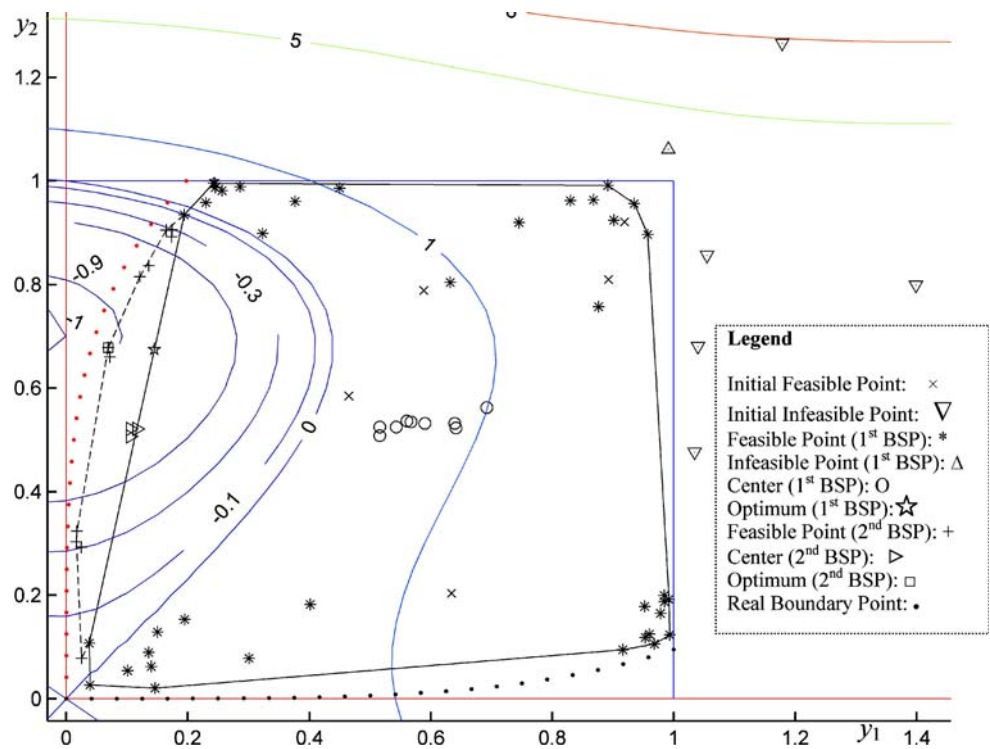
Fig. 10 Cumulative number of the SA/MDA over 16 BSIs of run no. 1 in test case 1
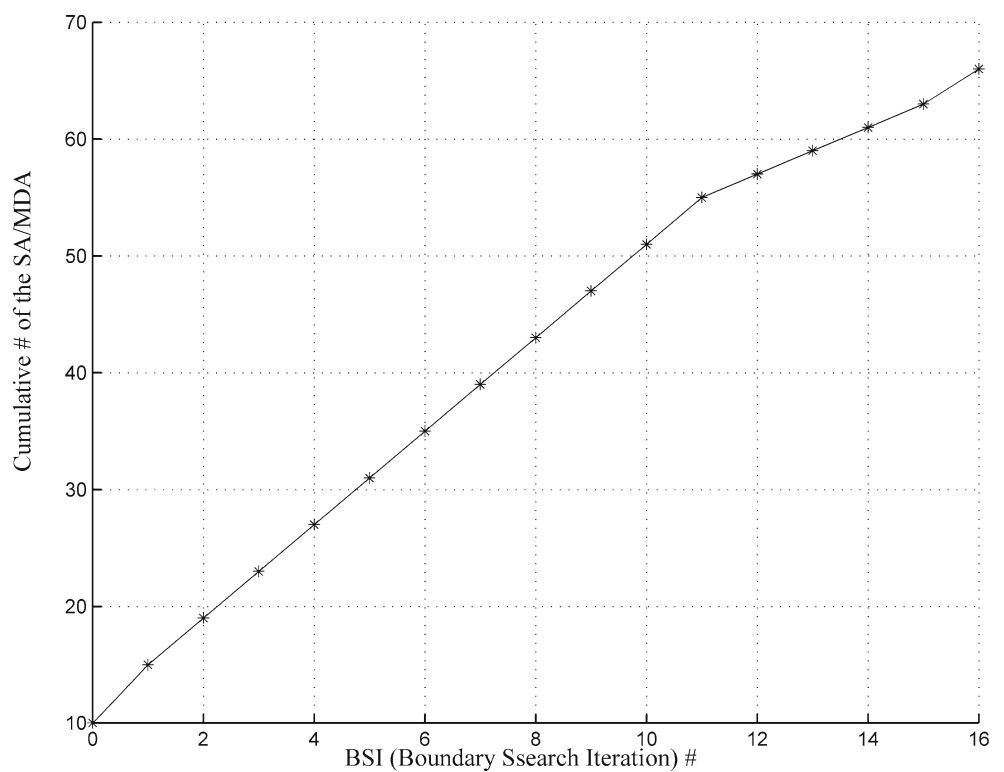
Fig. 11 Convergence history over 16 BSIs of run no. 1 in test case 1
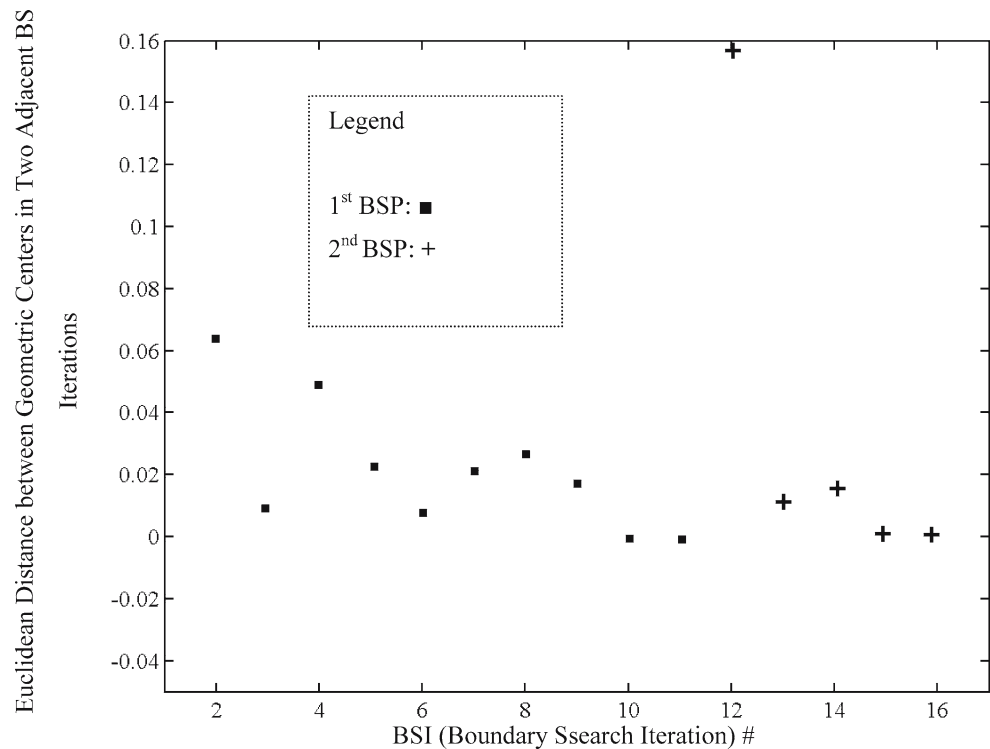
5.2 Test case 2

Taken from Tappeta et al. (1998), test case 2 only involves SA/MDA with the formulation as

$$
\begin{aligned}
y_1 &= x_1^2 + x_2 + x_3 - 4 - 0.2y_2 \\
y_2 &= x_1 + x_3 - 2 + \sqrt{y_1} \\
\text{subject to} &: 0 \le x_1 \le 7 \\
& \quad 2 \le x_2 \le 7 \\
& \quad 2 \le x_3 \le 7 \\
& \quad y_1/8 - 1 \ge 0 \\
& \quad 1 - y_2/10 \ge 0
\end{aligned}
\tag{28}
$$

The actual feasible region of y is represented by a limited number of dots, which were generated by calling the SA/MDA based on different combinations of design variables when given their ranges and intervals, as shown in Fig. 12. The solution of test case 2 given by the BSSDM, depicted in Figs. 12 and 13, takes 46 expensive feasible points and 21 expensive infeasible points, among which 11 are for initialization. The number of initial feasible expensive points is 5. Only one BSP is implemented. The BSSDM stopped after two consecutive BSIs because the geometric center ce does not move further. The cumulative number of the SA/MDA and the convergence curve are shown in Figs. 14 and 15, respectively. The result shows a good approximation of a star-like state parameter region.

5.3 Discussion

The BSSDM can robustly solve MDO problems with a feasible convex state parameter region. If the expense of solving the SA/MDA is of primary concern, when the feasible state parameter region is known, the rest of the work becomes inexpensive. The optimization process can be readily implemented by conventional optimization techniques, such as a gradient-based algorithm. Also, the feasible state parameter region shows a 'global' picture of the relation among coupled state parameters. This information gives designers freedom to choose a feasible design from the region. However, it currently cannot deal with a generic nonconvex region. The proposed star-like decomposition shows possibilities that a generic nonconvex region can be decomposed into local convex regions. Further research on this issue is needed.

It is observed that the CM is effective for coordinating coupled subsystems. The expensive sample points are inheritable because all expensive points are collectively used in the RBF models. Each call to the SA/MDA, regardless of its feasibility with respect to constraints, benefits the RBF approximations.

The proposed method is demonstrated effective for the two test cases, which could be conveniently illustrated. The method, however, is not limited to n=2. It is generally applicable to any number of state parameters. The computational expense, however, exponentially increases with the increase of the number of state parameters. It is fortunate that, compared with the number of disciplinary design

Fig. 12 Complete process information of the results of test case 2 given by the BSSDM
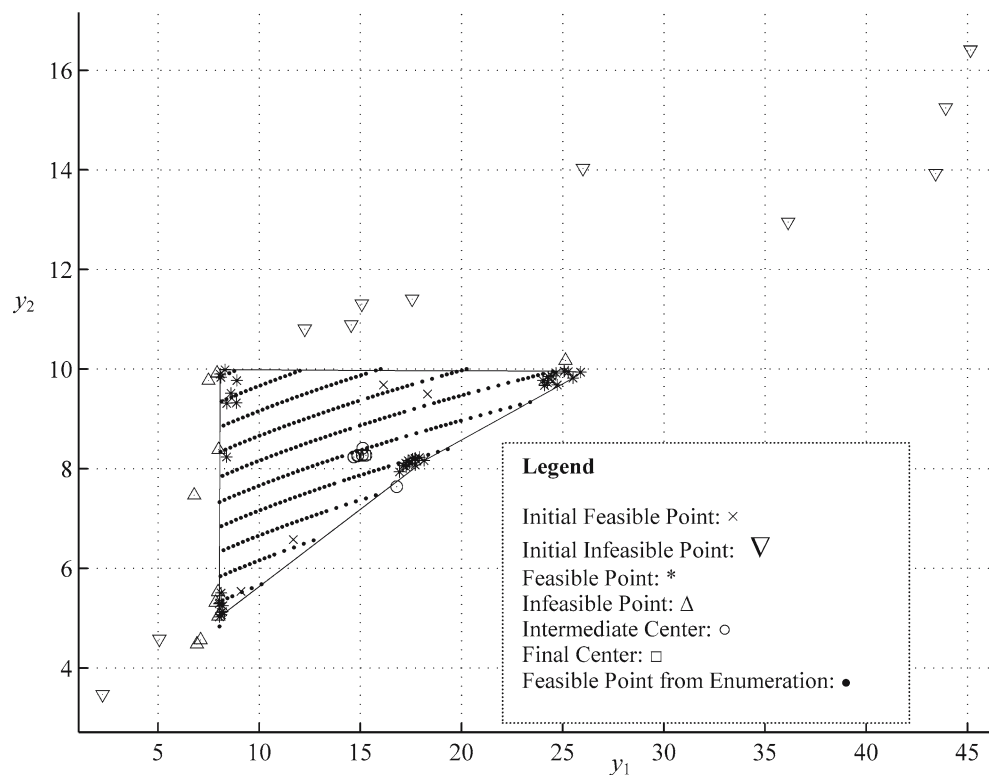
variables, x, the number of state parameters, n, is usually relatively small. For a pure convex state parameter region problem, it is observed that the boundary search converges quickly and no optimization method is required to explore the boundary in terms of the Euclidean distance. This could compensate the total computational cost for the boundary exploration of a large number of state parameters.

This work does not aim to develop a method that is competitive to the well-known MDO methods such as BLISS and CO. The purpose of this work is to provide a novel approach to better resolve the expensive coupling issue in MDO. It is hoped that the concept of the state parameter region and the BSSDM could lead to more innovative MDO approaches.

Fig. 13 Zoomed-in figure of the solution of y of test case 2 given by the BSSDM
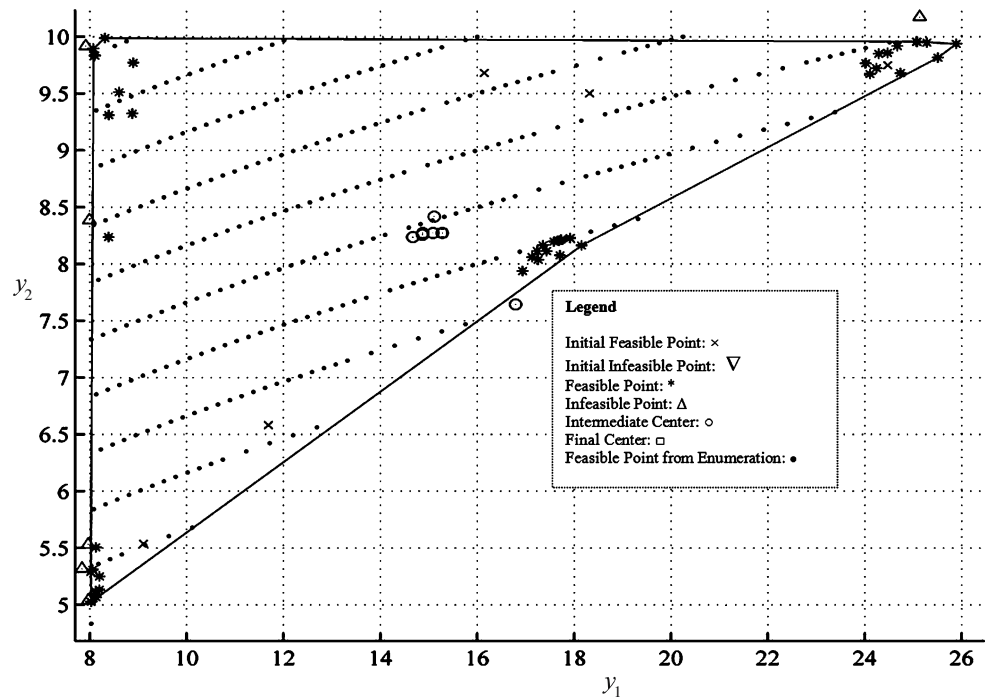
Fig. 14 Cumulative number of
the SA/MDA over 14 BSIs of
test case 2
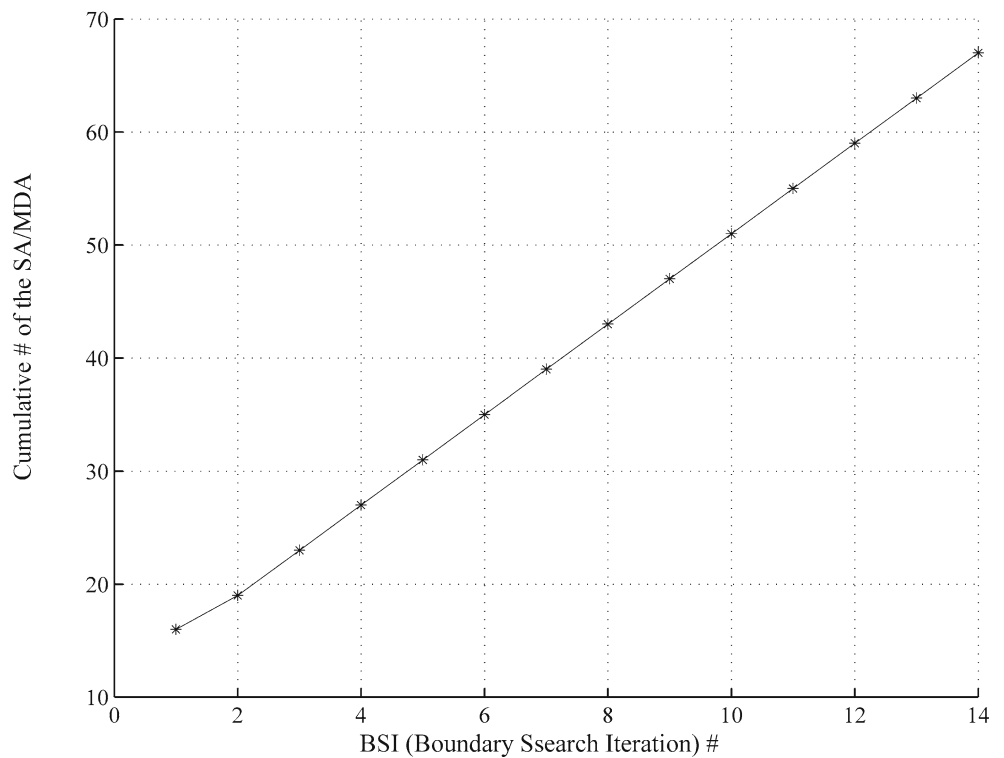
6 Concluding remarks

The couplings between disciplines characterize MDO
problems and remain a major computational challenge in
solving MDO problems. This paper proposes the concept of
a feasible state parameter region, which captures the
complex couplings among disciplinary analyses. Having
the information of the feasible state parameter region, the
expensive SA/MDA could be eliminated in search of the
system objective. Moreover, design engineers can choose
many design candidates that fall into the feasible region
and, thus, are freed up from the complex couplings among

Fig. 15 Convergence history
over 14 BSIs of test case 2
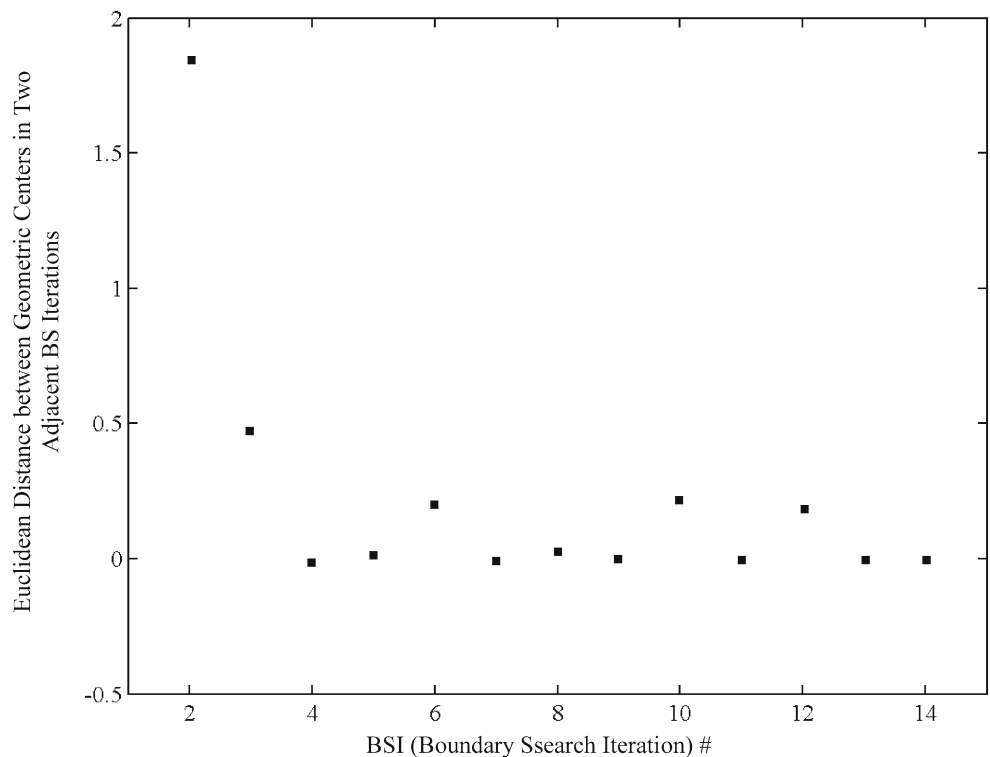
disciplines. The proposed BSSDM provides an innovative way to capture the feasible state parameter region. To the best of the authors' knowledge, it is the first time that MDO problems have been classified and solved in the context of the convex and nonconvex state parameter regions. The BSSDM, at its current embryonic stage, requires a specified MDO formation and is restricted to MDO problems with a convex or star-like state parameter region. The BSSDM is considered promising for future research for more generic MDO problems.

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