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Reliable Space Pursuing for Reliability-based Design Optimization with Black-box Performance Functions

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Abstract: Reliability-based design optimization (RBDO) is intrinsically a double-loop procedure since it involves an overall optimization and an iterative reliability assessment at each search point. Due to the double-loop procedure, the computational expense of RBDO is normally very high. Current RBDO research focuses on problems with explicitly expressed performance functions and readily available gradients. This paper addresses a more challenging type of RBDO problem in which the performance functions are computation intensive. These computation intensive functions are often considered as a "black-box" and their gradients are not available or not reliable. On the basis of the reliable design space (RDS) concept proposed earlier by the authors, this paper proposes a Reliable Space Pursuing (RSP) approach, in which RDS is first identified and then gradually refined while optimization is performed. It fundamentally avoids the nested optimization and probabilistic assessment loop. Three well known RBDO problems from the literature are used for testing and demonstrating the effectiveness of the proposed RSP method.

Key words: Reliability based design optimization, black-box function, reliable design space

Notations

$f(\cdot)$ —Objective function
$g_i(\cdot)$ —The *i*th constraint function
g_i^* —Boundary constraints of the reliable design space
k —Number of deterministic design variables
m —Number of random design variables
n —Number of constraints
q —Number of random parameters
$r_{\mathrm{d}i}$ —Desired reliability of satisfying the *i*th constraint
S_{r} —Reliable design space
u —Standard normal distribution space
v_j^u —Standard normal distribution variable
x —Deterministic variable or a realization of random variable X
$\boldsymbol{x}$ —Vector of deterministic variables
X —Random variable
$\boldsymbol{X}$ —Vector of random design variables, $\boldsymbol{X} \in \mathbf{R}^m$
α_j —Direction cosine along the axis x_j
$\beta_{\mathrm{d}i}$ —Index of the desired reliability of satisfying the *i*th constraint
$\beta_{\mathrm{s}i}$ —Index of the success probability of satisfying the *i*th constraint
$\left(\dfrac{\partial g_i}{\partial \overline{x}_j^u}\right)_*$ —Partial derivative at the most probable point (MPP) in the standard normal distribution space
d^{Lower} —Lower limit of deterministic design variables
$\mathrm{d}^{\mathrm{Upper}}$ —Upper limit of deterministic design variables
$\boldsymbol{d}$ —Vector of deterministic design variables, $\boldsymbol{d} \le \mathbf{R}^k$
$F_{X,P}(\boldsymbol{X}, \boldsymbol{P})$ —Joint probability density function of all random variables and random parameters
$Prob(\cdot)$ —Probability function
$\boldsymbol{P}$ —Vector of random parameters, $\boldsymbol{P} \le \mathbf{R}^q$
$\boldsymbol{V}$ —Vector of random design variables and random parameters combined $(\boldsymbol{X}, \boldsymbol{P})$
$\boldsymbol{\mu_X}$ —Mean vector of $\boldsymbol{X}$
$\boldsymbol{\mu_P}$ —Mean vector of $\boldsymbol{P}$
$\boldsymbol{\mu_V}$ —Mean vector of $\boldsymbol{V}$
$\boldsymbol{\sigma_X}$ —Standard deviation vector of $\boldsymbol{X}$
$\boldsymbol{\sigma_P}$ —Standard deviation vector of $\boldsymbol{P}$
$\boldsymbol{\sigma_V}$ —Standard deviation vector of $\boldsymbol{V}$
Φ^{-1} —Inverse transformation of Φ
Φ —Standard normal distribution function
$*$ —Values evaluated at MPP

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1 Introduction

Reliability-based design optimization (RBDO) is a new design methodology to optimize a product's reliability with respect to various uncertainties. One of the most challenging issues for implementing RBDO is related with the intensive computational demand of the reliability assessment within the optimization process.

For the purpose of improving computational efficiency, ANG, et al[1], introduced the reliability index approach (RIA) to describe the probability and to simplify the reliability analysis. A performance measure approach (PMA) was proposed in Ref. [2] to enhance probabilistic constraint evaluation in RBDO. PMA was then improved to PMA+ [3] and the applications of PMA were in Refs. [4−5]. DU, et al[6], developed a sequential strategy with a series of sequential deterministic optimization and reliability assessment processes. LIANG, et al[7], presented a single-loop RBDO algorithm in which the reliability at the current point was approximated by using the information at the previous search point. WU, et al[8], converted reliability constraints to approximately- equivalent deterministic constraints, based on which a safety-factor based approach was developed [9]. YANG, et al[10], implemented and tested several approximate RBDO methods against a double loop algorithm with a number of design problems. SHAN, et al[11], developed a novel concept of reliable design space (RDS) within which every design point satisfies the reliability requirements, and proposed an analytic single loop RBDO approach by writing out the boundaries of RDS, which enables RBDO to be solved with any optimizer. This approach is suitable for RBDO problems with explicit objective and constraint functions. In contrast to most current methods, this approach follows an inverse procedure, i.e., the reliable space is identified before the optimization starts. The concept of RDS is illustrated in Fig. 1 by using a 2D case.

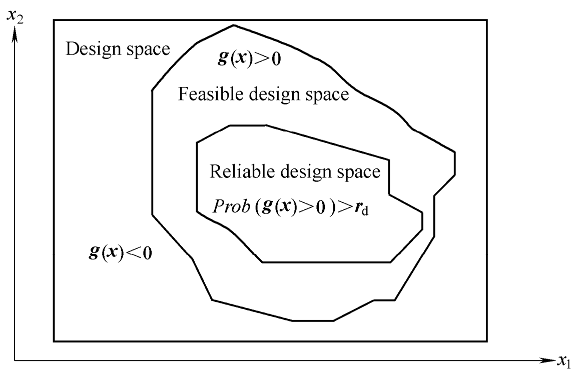

Fig. 1. Concept of the reliable design space

In Fig. 1 there exist three spaces, i.e., the design space, feasible design space, and reliable design space. The design space is represented by the outer rectangle. The feasible design space is separated from the design space by the deterministic constraints $\boldsymbol{g}(\boldsymbol{x})=\boldsymbol{0}$ without considering probability and is the subset of the design space. The reliable design space is formed by the probability constraints $Prob(\boldsymbol{g}(\boldsymbol{x})>\boldsymbol{0})>\boldsymbol{r}_{\mathrm{d}}$ and is the subset of the feasible design space. Where $Prob(\bullet)$ is the probability function that denotes the probability of satisfying a vector of constraints $\boldsymbol{g}(\boldsymbol{x})>\boldsymbol{0}$. $\boldsymbol{r}_{\mathrm{d}}$ is the vector of the desired reliability of satisfying the constraints. If the reliable design space can be identified before the optimization process, the inner reliability assessment loop of RBDO can be eliminated because every point in the reliable design space meets the reliability requirement. In other words, the optimization process is constrained by the boundaries of the reliable design space. Then, the RBDO problem becomes a simple deterministic optimization problem constrained by the boundaries of RDS.

Ref. [11] dealt with inexpensive performance functions for which gradients were readily available, where the boundaries of RDS could be expressed explicitly. As an extension of Ref. [11], this paper addresses RBDO problems with expensive performance functions whose gradients are not available (either because the computation expense is too high or the gradients cannot be accurately computed). Related theories will be first introduced in the next section. In section 3, the proposed methodology is described. A few well known problems from the literature are used for testing the proposed method and the test results are given in section 4. Section 5 is the conclusion.

2 Related Concepts and Theories

In this section, we first introduce the concept of reliability-based design optimization, and then discuss the kriging model that is applied to model the constraint functions. Finally the mode pursuing method (MPS) will be briefly introduced as it is used for optimization in this work.

2.1 Concept of reliability-based design optimization

A typical RBDO problem is formulated as follows:

$$\begin{cases} \min\limits_{d,\mu_X} f(\boldsymbol{d}, \boldsymbol{\mu_X}, \boldsymbol{\mu_P}), \\ \text{s.t.} \quad Prob(g_i(\boldsymbol{d}, \boldsymbol{X}, \boldsymbol{P})>0)>r_{\mathrm{d}i}, i=1,\cdots,n, \\ \boldsymbol{d}^{\mathrm{Lower}} \le \boldsymbol{d} \le \boldsymbol{d}^{\mathrm{Upper}}, \boldsymbol{\mu}_{\boldsymbol{X}}^{\mathrm{Lower}} \le \boldsymbol{\mu_X} \le \boldsymbol{\mu}_{\boldsymbol{X}}^{\mathrm{Upper}}. \end{cases} \quad (1)$$

Where superscripts "Lower" and "Upper" denote the lower and upper limits, respectively. As per the traditional notation, a bold letter indicates a vector, an upper case letter indicates a random variable or a random parameter and a lower case letter indicates a realization of a random variable or random parameter.

Eq. (1) shows that RBDO involves a so-called double-loop procedure where the optimization outer loop includes inner loops of reliability analysis. The inner loop or reliability analysis is often treated as an optimization problem searching for the most probable point (MPP) [1], a concept used for reliability assessment. The iterative search of MPP accounts for the main computation expense for RBDO, which often makes the double-loop strategy computationally challenging. In order to clearly describe the nature of RBDO and reduce the computational burden of RBDO, Ref. [11] proposed the reliable design space (RDS) concept. After the reliability design space has been introduced, the RBDO problem in Eq. (1) can be converted into a deterministic optimization problem by means of the reliable design space as follows:

$$\begin{cases} \min\limits_{\boldsymbol{d},\,\mu_X} f(\boldsymbol{d},\boldsymbol{\mu_X},\boldsymbol{\mu_P}), \\ \text{s.t.}\quad g_i^*(\boldsymbol{d},\boldsymbol{\mu_X},\boldsymbol{\mu_P})>0,\ i=1,\cdots,n, \\ \boldsymbol{d}^{\text{Lower}}\leqslant \boldsymbol{d}\leqslant \boldsymbol{d}^{\text{Upper}}, \\ \boldsymbol{\mu}_{\boldsymbol{X}}^{\text{Lower}}\leqslant \boldsymbol{\mu_X}\leqslant \boldsymbol{\mu}_{\boldsymbol{X}}^{\text{Upper}}. \end{cases} \tag{2}$$

Given the definition of the reliability design space, the next question is how to find this reliable design space. Ref. [11] clearly described the key equation of RBDO which formulates the relation between a design point and its MPP in the X-space, as shown in Eq. (3):

$$\bar{\mu}_{x_j}=\bar{x}_j^*+\beta_{\mathrm{d}i}\bar{\sigma}_{x_j}^2\frac{\left(\dfrac{\partial g_i}{\partial \bar{x}_j}\right)_*}{\sqrt{\sum\limits_j\left(\bar{\sigma}_{x_j}\dfrac{\partial g_i}{\partial \bar{x}_j}\right)_*^2}}, \tag{3}$$

where * denotes the MPP in the original design space, which is often referred as the inverse MPP[6]. The vector $\boldsymbol{\mu}_{\bar{X}}$ is the design point and $\bar{\boldsymbol{x}}^*$ is its inverse MPP. $\beta_{\mathrm{d}i}$ is the index of the desired reliability of the ith constraint function, and $\bar{\sigma}_{x_j}$ is the standard deviation for x_j. The vector $\bar{\boldsymbol{X}}=(\boldsymbol{X},\boldsymbol{P})^{\mathrm{T}}=(x_1,x_2,\cdots,x_m,p_1,p_2,\cdots,p_q)^{\mathrm{T}}$ includes all random variables and parameters (refer to Fig. 2 for an illustration of a two-variable problem with one constraint $g(x_1, x_2)$).

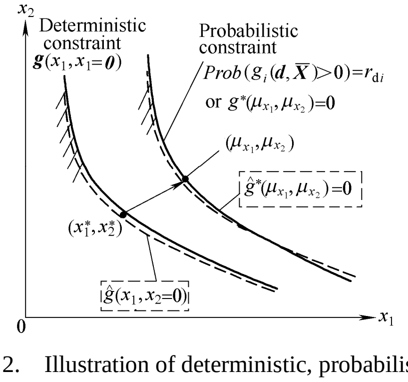

Fig. 2. Illustration of deterministic, probabilistic, and approximated constraints

As revealed in Ref. [6], the evaluation of $Prob(g(\boldsymbol{d},\bar{\boldsymbol{X}})>0)=r_{\mathrm{d}}$ at design point $\boldsymbol{\mu}_{\bar{X}}$ is equivalent to evaluating the deterministic constraint at the inverse MPP, $g(\boldsymbol{d},\bar{X}^*)=0$. The essential task, as well as the fundamental challenge, of RBDO is therefore to find a design point $\boldsymbol{\mu}_{\bar{X}}$, whose corresponding inverse MPP is within the deterministic feasible domain. In RBDO procedure, the outer optimization loop updates the design point $\boldsymbol{\mu}_{\bar{X}}$ at every iteration. The corresponding inverse MPP, however, is not easy to find by directly using Eq. (3). An iterative numerical process is required, either be it optimization or solving simultaneous equations, which forms the so-called inner-loop for reliability assessment. Recent research on RBDO focuses on this very issue by proposing approximation or iterative methods to avoid or reduce the effort in solving Eq. (3) when given a design point $\boldsymbol{\mu}_{\bar{X}}$.

In this work all constraint functions, or performance functions, are assumed to be black-box functions whose gradients are not available. It is thus impossible to directly solve Eq. (3) at each design point $\boldsymbol{\mu}_{\bar{X}}$. The proposed approach applies metamodeling to approximate the limit state function, $g_i(x)=0$, and its partial derivates, $\partial g_i/\partial x_j$, on the deterministic feasible space boundary. Then we use Eq. (3) to obtain functions of $\boldsymbol{\mu}_{\bar{X}}$, denoted by function $g_i^*=0$, which form the boundaries of the reliable design space. Then a deterministic optimization problem can be formed as described in Eq. (2), which is used for locating the optima and in the mean time improve the accuracy of the metamodel. The kriging model is chosen in this work for the metamodel.

2.2 Kriging model

For a computational intensive problem or black-box function, metamodeling is commonly used to approximate the expensive or black-box function. The metamodel chosen to construct the constraint functions in this work is the kriging model as defined below [12−14]:

$$\hat{y}(X)=\sum_{i=1}^{k}\alpha_i f_i(X)+z(X). \tag{4}$$

Kriging model consists of two parts. The first part is a simple linear regression of the data. The second part is a random process. The coefficients, α_i, are regression parameters; $f_i(x)$ is the regression model. The random process $z(X)$ is assumed to have mean zero and covariance, $v(x_i,x_j)=\sigma^2R(x_i,x_j)$. The process variance is given by σ^2 and its standard deviation is σ. The smoothness of the model, the influence of other nearby points, and differentiability of the response surface are controlled by the spatial correlation function, $R(\bullet)$. Kriging is flexible to approximate different and complex response functions. The response surface of kriging model interpolates sample points, and the influence of other nearby points is controlled by the spatial correlation function. On the basis of these features, the kriging model is chosen in this work.

A kriging toolbox is given by Ref. [15]. It provides regression models with polynomials of orders 0, 1, and 2, as well as 7 spatial correlation functions for selection. This work uses the regression model with polynomials of order 0, and the Gaussian correlation model. A detailed description of kriging is in the corresponding author's previous work [13].

2.3 Mode pursing sampling method

This work uses the mode pursing sampling (MPS) method for the optimization constrained by RDS, although other standard optimization routines are equally acceptable for the proposed method. MPS is a type of statistic sampling optimization method[16]. MPS generates more sample points in areas having lower objective function value and fewer points in other areas. MPS first constructs an approximation model from a few sample points. It then generates a large number of points from the approximation model, sorts the points, and constructs a cumulative function analogous to cumulative density function (CDF) by adding up all the function values listed before the current point in the sorted point set. A sample is then drawn from the point set according to this cumulative function using the inverse CDF sampling method. As a result, more new sample points are generated around the current minimum and less in other regions in a design space. MPS is an iterative process and the optimum is found as the sampling process proceeds. MPS is in essence a discriminative sampling method with approved robustness and convergence property.

3 Proposed Reliable Space Pursuing Methodology

Given the concept of RDS, the proposed reliable space pursuing (RSP) method is to identify boundaries of RDS within the design space, based on which optimization is performed. The proposed methodology has two stages. The first stage is to construct metamodels of the constraint functions so that RDS boundaries can be approximated. The second stage is an optimization process with improvements to the metamodels of RDS boundaries.

3.1 Approximating RDS boundaries

Boundaries of RDS, $Prob(\boldsymbol{g}(\boldsymbol{X})>\boldsymbol{0})=r_{\mathrm{d}}$ or its equivalent $g_i^*=0$, are the probabilistic constraints, while $g_i=0$ are the deterministic constraints, or the boundaries of the feasible space. For the ease of description, we assume two random variables X_1 and X_2 are involved in a RBDO problem and there is no random parameter $\boldsymbol{P}$. The metamodeling process can be conceptually illustrated by Fig. 2.

Fig. 2 shows both the deterministic constraints $\boldsymbol{g}(x_1, x_2)=\boldsymbol{0}$ and probabilistic constraints $\boldsymbol{g}^*(\mu_{x_1}, \mu_{x_2})=\boldsymbol{0}$. The dotted lines show their corresponding metamodels to be built, $\hat{\boldsymbol{g}}(x_1, x_2)=\boldsymbol{0}$ and $\hat{\boldsymbol{g}}^*(\mu_{x_1}, \mu_{x_2})=\boldsymbol{0}$, respectively. The metamodeling process first generates points around $\boldsymbol{g}(x_1, x_2)=\boldsymbol{0}$ to build $\hat{\boldsymbol{g}}(x_1, x_2)=\boldsymbol{0}$. The process iterates until it converges with a reasonably accurate $\hat{\boldsymbol{g}}(x_1, x_2)=\boldsymbol{0}$. Then Eq. (3) is called to generate points on $\boldsymbol{g}^*(\mu_{x_1}, \mu_{x_2})=\boldsymbol{0}$ where the gradients in Eq. (3) are calculated by the finite difference method on $\hat{\boldsymbol{g}}(x_1, x_2)=\boldsymbol{0}$. The model $\hat{\boldsymbol{g}}^*(\mu_{x_1}, \mu_{x_2})=\boldsymbol{0}$ is then constructed using the calculated points from $\hat{\boldsymbol{g}}(x_1, x_2)=\boldsymbol{0}$.

In specific, the steps at this approximation stage are described below. For clarity, points evaluated by calling expensive constraint functions are referred as evaluated points or expensive points. In contrast, points calculated from metamodels are referred as cheap points.

(1) Initially sampling a few points in the design space and evaluating all deterministic constraint functions at these points (expensive points).

(2) Building an approximation model for each deterministic constraint using the evaluated points.

(3) Generating a large number of points, for example, 10 000, in the design space and predicting values of these points by the approximation models (cheap points).

(4) Filtering out points that cannot satisfy all deterministic constraints as predicted by the approximation models and keeping the rest of the points.

(5) For each constraint, sorting the leftover points in an ascending order according to their predicted function values, constructing a cumulative function analogous to CDF by adding up all the function values listed before the current point in the sorted point set. New samples are drawn from the point set according to this cumulative function. The sampling process is discriminative as that in MPS [16] and the authors' other work [17−19]. As a result, more new sample points are around the boundaries of the feasible space $\boldsymbol{g}(x_1, x_2)=\boldsymbol{0}$ and less in other regions in the feasible design space.

(6) Evaluating the new sampling points from step (5) and these points become expensive points.

(7) Checking the convergence of metamodels.

(8) Updating the metamodels.

(9) If none of the convergence condition is reached, back to step (2) with the new expensive points. Otherwise temporarily end up the metamodeling stage and enter the optimization stage. There are two convergence criteria. The first convergence criterion is

$$\max\left(\frac{|g_i(x_1, x_2)-\hat{g}_i(x_1, x_2)|}{|g_i(x_1, x_2)|}\right)\leqslant 0.000\,1$$

on the current new sample points for all constraints. The second convergence criterion is the maximum number of metamodeling iterations, e.g., 50. Once either of the two criteria is satisfied for all constraints, the metamodeling stage temporarily ends. These two criteria can be adjusted according to specific needs.

(10) Generating cheap points from $\hat{\boldsymbol{g}}(x_1, x_2) = \boldsymbol{0}$, using Eq. (3) to compute points on $\boldsymbol{g}^*(\mu_{x_1}, \mu_{x_2}) = \boldsymbol{0}$. With the computed points, we can use the kriging model to build $\hat{\boldsymbol{g}}^*(\mu_{x_1}, \mu_{x_2}) = \boldsymbol{0}$ for optimization in the next stage.

Recognizing that the metamodel $\hat{g}(x_1, x_2)$ will inevitably have errors as compared to the true expensive performance function $g(x_1, x_2)$, and so its gradients, this work introduces a feedback step at the second stage. That is, assuming $\hat{\boldsymbol{g}}^*(\mu_{x_1}, \mu_{x_2}) = \boldsymbol{0}$ calculated from Eq. (3) by using $\hat{g}(x_1, x_2)$ and its gradients has certain error, the obtained optimum (usually constrained optimum) will be evaluated by calculating its corresponding inverse MPP to see whether the MPP satisfies $\boldsymbol{g}^*>0$. If the optimum and corresponding inverse MPP fail to satisfy the actual constraints due to the use of metamodels, this evaluated optimum (an expensive point) and corresponding inverse MPP (an expensive point) will be added to the existing set of expensive points and the metamodels are to be updated. Therefore, the accuracy of metamodels can be further improved. Such a step provides a degree of error compensation and the compensation is well observed in testing problem 2, which will be described in section 4.2.

3.2 Optimization with metamodels of probabilistic constraints

Once the metamodels of probabilistic constraints are constructed, we use the metamodels as surrogates for the actual constraints in Eq. (2) and apply the MPS method in Ref. [16] to solve the optimization problem. As discussed in the last section, the obtained optimum and corresponding inverse MPP will be verified by calling the actual expensive constraint functions. The flowchart of the proposed method is illustrated in Fig. 3. The two major stages are enclosed in a dotted box, respectively. Special attentions need to be paid to the expressions of metamodels for different constraint functions.

4 Numerical Studies

In this section, three problems are employed to test the proposed methodology. We assume the constraint functions are computation intensive and thus treated as black-box functions. For each problem, ten independent runs are carried out to test the robustness of the proposed method.

4.1 Problem 1

Problem 1 was first introduced in Ref. [20] and then used by others [7, 10]. It has two random variables X_1, X_2, which are normally distributed, and three non-linear constraints, g_1, g_2 and g_3. There is no deterministic design variable and no random parameter. The objective function is simply the sum of the mean of the two random variables. The RBDO problem is described as follows:

$$\begin{cases} \min\limits_{\mu_1,\mu_2} f = \mu_1 + \mu_2, \\ \text{s.t. } P(g_i(X) \geq 0) \geq R_i,\ i = 1-3, \\ \quad g_1(X) = \dfrac{X_1^2 X_2}{20} - 1, \\ \quad g_2(X) = \dfrac{(X_1 + X_2 - 5)^2}{30} + \dfrac{(X_1 - X_2 - 12)^2}{120} - 1, \\ \quad g_3(X) = \dfrac{80}{X_1^2 + 8X_2 + 5} - 1, \\ \quad 0 \leq \mu_j \leq 10,\ j = 1-2, \\ \quad \sigma_1 = \sigma_2 = 0.3, \\ \quad \beta_i = 3 \text{ for } i = 1, 2, 3. \end{cases} \tag{5}$$

Where μ_1, μ_2 and σ_1, σ_2 are the mean values and standard deviations, respectively, of the two design random variables X_1 and X_2. R_i is the target reliability of ith constraint. For demonstration purposes, the same target reliability index $\beta = 3$ is used for all three constraints. In general, a different target reliability index may be used for each constraint.

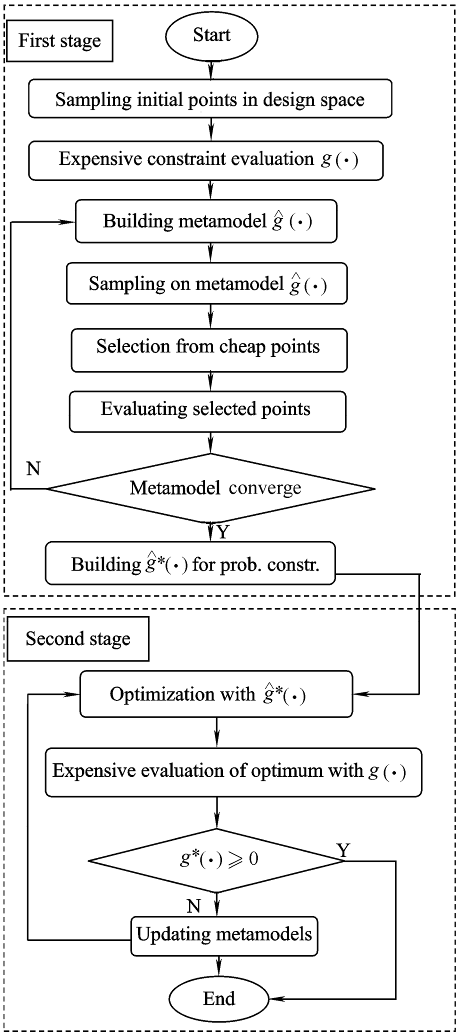

Fig. 3. Flowchart of the proposed methodology

By applying the proposed methodology, the obtained results are listed in Table 1. Columns $g_i(x)$ and $g_i^*(x)$ represent the deterministic and corresponding probabilistic constraint function values, respectively. The number of iterations, n_{it}, and the number of function evaluations, n_{fe}, are used as an indication of the time and resources required in the computation for expensive function evaluations. The number of function evaluations, n_{fe}, for each constraint includes all evaluations in two stages (n_{fe} is the same for all constraints). As can be seen from Table 1, the proposed method robustly captures the same RBDO optimum with ten independent runs, which is almost identical to the theoretical optimum at (3.44, 3.28) with the minimum objective function value 6.720 5 [11]. The computational expense, as indicated by n_{fe}, is modest.

Table 1. Test results for problem 1

Run No.	μ_X^*	$f(\mu_X^*)$	g_i			g_i^*			n_{it}		n_{fe}	
			g_1	g_2	g_3	g_1^*	g_2^*	g_3^*	First stage	Second stage	f	g_i
1	[3.440 6, 3.280 0]	6.720 5	0.000 0	0.000 0	0.511 7	0	0	>0	6	2	13	45
2	[3.440 5, 3.280 0]	6.720 5	0.000 0	0.000 0	0.511 7	0	0	>0	4	3	16	38
3	[3.440 6, 3.280 0]	6.720 5	0.000 0	0.000 0	0.511 7	0	0	>0	7	3	16	55
4	[3.440 6, 3.280 0]	6.720 5	0.000 0	0.000 0	0.511 8	0	0	>0	4	3	16	37
5	[3.440 6, 3.280 0]	6.720 5	0.000 0	0.000 0	0.511 8	0	0	>0	9	2	12	64
6	[3.440 6, 3.280 0]	6.720 5	0.000 0	0.000 0	0.511 8	0	0	>0	5	3	15	43
7	[3.440 6, 3.280 0]	6.720 5	0.000 0	0.000 0	0.511 8	0	0	>0	8	2	12	59
8	[3.440 6, 3.280 0]	6.720 5	0.000 0	0.000 0	0.511 8	0	0	>0	9	2	12	65
9	[3.440 6, 3.280 0]	6.720 5	0.000 0	0.000 0	0.511 8	0	0	>0	6	3	15	50
10	[3.440 6, 3.280 0]	6.720 5	0.000 0	0.000 0	0.511 8	0	0	>0	10	2	12	71

Fig. 4 plots the identified reliable design space by using the proposed method, shown as the shaded area. The shaded area is enclosed by the theoretical RDS boundaries plotted with “×” symbols. It indicates that the reliable design space has been identified accurately. The dot “◆” represents the evaluated expensive points for metamodeling. The circle “○” represents the evaluated expensive points during the optimization stage, which are used to update the metamodels. These points fall inside the reliable design space.

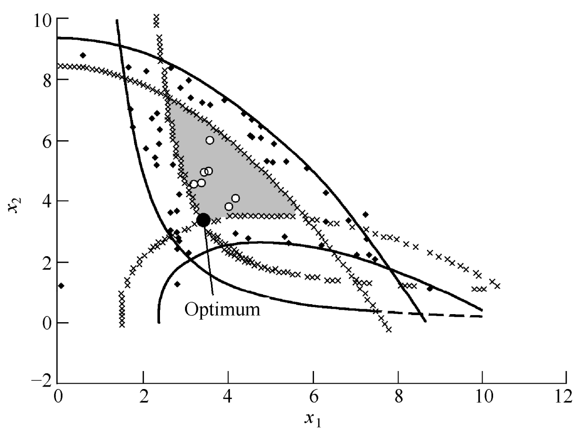

Fig. 4. Graphical output of test results for problem 1

4.2 Problem 2

A cantilever beam in vertical and lateral bending was used in Refs. [2−5, 9, 21]. The beam is loaded at its tip by the vertical and lateral loads F_Y and F_Z, respectively. Its length L is equal to 100 in (2.54 m). The width w and thickness t of the cross-section are random design variables. The objective is to minimize the weight of the beam. This is equivalent to minimizing $f = wt$, assuming that the material density and beam length are constant. Two non-linear failure modes are used. The first failure mode yields at the fixed end of the cantilever; the other failure mode is that the tip displacement exceeds the allowable value $D_0 = 2.5$ in (63.5 mm). The RBDO problem is formulated as follows:

$$\begin{cases} \min\limits_{w,t} f = wt, \\ \text{s.t.} \quad P(g_i \geq 0) \geq R_j,\ i = 1-2, \\ \quad g_1(S, F_Z, F_Y, w, t) = S\left(\dfrac{600}{wt^2}F_Y + \dfrac{600}{w^2 t}F_Z\right), \\ \quad g_2(E, F_Z, F_Y, w, t) = D_0 \dfrac{4L^3}{Ewt} \bullet \\ \quad \sqrt{\left(\dfrac{F_Y}{t^2}\right)^2 + \left(\dfrac{F_Z}{w^2}\right)^2}, \\ \quad 1 \leq w, t \leq 5. \end{cases} \tag{6}$$

Where g_1 and g_2 are the two constraints corresponding to the two failure modes. The design variables w and t are deterministic, while F_Y, F_Z, S and E are normally distributed random parameters with $F_Y \sim N(1\,000, 100)\,lb, F_Z \sim N(500, 100)\,lb$, $S \sim N(40\,000, 2\,000)\,psi$ and $E \sim N(29\times10^6, 1.45\times10^6)\,psi$. The parameter y is the random yield strength; F_Z and F_Y are mutually independent random loads in the vertical and lateral directions respectively, and E is the Young’s modulus. R_i is the target reliability of the ith constraint. A reliability index $\beta = 3$ is used for both constraints. In this case, only two deterministic design variables exist in the objective function, and only these deterministic design variables and four random parameters exist in the probability constraints. No random design variable exists in either the objective or probability constraint functions. Test results are listed in Table 2 in a format similar to Table 1. One can see that the proposed method consistently reaches almost the theoretical optimum at (2.45, 3.89) with the minimum function value 9.52[11]. The number of evaluation of the constraints, however, increases as compared to problem 1 due to the complex form of constraints of problem 2. A similar graphical plot is also generated as shown in Fig. 5 (the symbols meaning is the same to those in Fig. 4). As one can see, the two theoretic probabilistic constraints are in a certain distance from their respective deterministic constraints. The shaded area is identified as the reliable design space by the proposed method. This area is not entirely enclosed by the theoretical probabilistic constraints. This indicates the error of the metamodels. Through iterations, it is observed that the initially obtained optimum does not satisfy the reliability constraint. However, the metamodel is gradually improved according to the algorithm and the final optimum satisfies the constraints and is close to the theoretical optimum.

Table 2. Test results for problem 2

Run No.	$\boldsymbol{\mu}_X^*$	$f(\mu_X^*)$	g		g_i^*		n_{it}		n_{fe}	
			g_1	g_2	g_1^*	g_2^*	First stage	Second stage	f	g_i
1	[2.457 1, 3.871 5]	9.512 9	0.825 3	0.301 1	0.000 0	0.310 2	72	6	64	320
2	[2.439 2, 3.902 5]	9.518 9	0.713 5	0.257 7	0.000 8	0.254 3	160	8	76	673
3	[2.385 9, 3.981 8]	9.500 4	1.332 3	0.233 3	0.000 0	0.310 2	100	4	53	427
4	[2.450 7, 3.884 4]	9.519 3	1.184 4	0.260 3	0.000 0	0.310 2	146	6	65	610
5	[2.450 7, 3.884 4]	9.519 3	1.184 4	0.260 3	0.000 0	0.310 2	146	6	65	610
6	[2.465 6, 3.860 9]	9.519 4	0.713 5	0.275 9	0.000 8	0.252 2	146	6	64	610
7	[2.377 6, 4.004 6]	9.521 2	9.112 9	0.196 8	0.000 9	0.280 8	44	5	56	206
8	[2.411 8, 3.940 8]	9.504 5	1.606 8	0.260 4	0.000 9	0.240 3	115	6	69	489
9	[2.461 1, 3.861 7]	9.504 1	0.803 7	0.315 7	0.000 0	0.310 2	82	8	80	365
10	[2.449 5, 3.882 8]	9.515 9	0.707 9	0.254 4	0.000 8	0.250 0	103	7	74	465

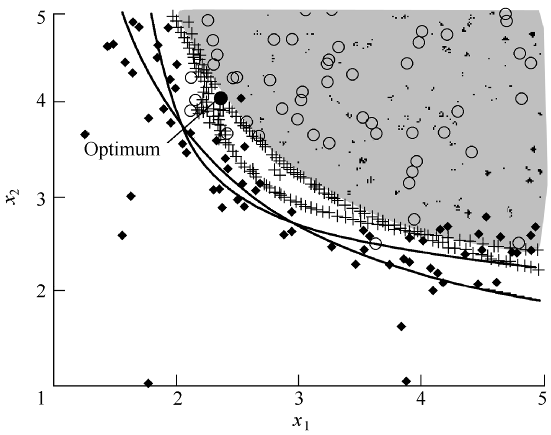

Fig. 5. Graphical output of test results for problem 2

4.3 Problem 3

A vehicle crashworthiness study has been extensively used to test the accuracy and efficiency of RBDO methods in the past a few years [4,6–7, 10]. Regarding to its engineering background, please refer to the references. Here omitting the engineering background, the RBDO vehicle model for crashworthiness is given as follows:

$$\begin{cases} \min\limits_{\boldsymbol{\mu}_x} f(\boldsymbol{\mu}_x, \boldsymbol{\mu}_p) = 1.98 + 4.90\mu_{x_1} + 6.67\mu_{x_2} + \\ \qquad 6.98\mu_{x_3} + 4.01\mu_{x_4} + 1.78\mu_{x_5} + \\ \qquad 2.73\mu_{x_7}, \\ \text{s.t.} \quad g_i(\boldsymbol{X}) \geq R_i,\ i = 1-10. \end{cases} \tag{7}$$

Where $g_i(\bullet)$ functions are listed in Appendix; R_i are the target reliabilities for each constraint; the 99.87% reliability ($\beta = 3$) is used for all ten constraints. As shown in Table 3, the reliability-based design optimization increases slightly the vehicle weight to approximate 28.6 for ($\beta = 3$) but satisfies all probabilistic constraints with at least 99.87% reliability. The proposed method robustly captures the same RBDO optimum with ten independent runs. The number of iterations, n_{it}, and the number of the function evaluations, n_{fe}, are used as an indication of the time and resources required in the computation for expensive function evaluations, which are found being modest.

5 Conclusions

(1) This work presents a reliability-based design optimization approach for problems involving expensive performance functions, for which the gradients of constraints are expensive to obtain or unreliable.

(2) The proposed method, RSP, directly approximates RDS through the inherent relationship between the deterministic and probabilistic constraints.

(3)Through the numeral tests, the RSP approach is found to be effective and robust. Its efficiency is affected by the complexity of the performance function.

(4) The proposed method, as an MPP-based approach, may have difficulties with highly-nonlinear constraints where multiple MPP may exist.

(5) The kriging modeling approach is also limited to small scale design problems due to its high cost and demand for exponentially increasing number of sample points.

Table 3. Test results for problem 3

Run No.	$\boldsymbol{\mu}_X^*$	$f(\boldsymbol{\mu}_X^*)$	g_i	g_i^*	n_{it}		n_{fe}	
					First stage	Second stage	f	g_i
1	(0.801 8 1.350 0 0.714 7 1.500 0 0.875 0 1.200 0 0.400 0 0.345 0 0.192 0 0 0)	28.566 4	(0.436 4 0.000 0 4.335 9 3.591 5 0.090 0 0.090 0 0.027 5 0.000 0 0.557 7 0.013 3)	(0.526 4 0.000 2 1.623 1 1.872 4 0.076 5 0.072 9 0.016 2 0.000 0 0.380 6 0.317 8)	51	7	140	1 030
2	(0.800 8 1.350 0 0.715 2 1.500 0 0.875 0 1.200 0 0.400 0 0.345 0 0.192 0 0 0)	28.564 9	(0.436 4 0.000 0 4.335 9 3.591 5 0.090 0 0.090 0 0.027 5 0.000 0 0.557 7 0.013 3)	(0.526 6 0.000 2 1.615 9 1.865 6 0.076 6 0.072 8 0.016 3 0.000 1 0.379 0 0.288 8)	51	7	150	1 035
3	(0.801 8 1.350 0 0.714 7 1.500 0 0.875 0 1.200 0 0.400 0 0.345 0 0.192 0 0 0)	28.566 4	(0.436 4 0.000 0 4.335 9 3.591 5 0.090 0 0.090 0 0.027 5 0.000 0 0.557 7 0.013 3]	(0.526 4 0.000 2 1.623 1 1.872 4 0.076 5 0.072 9 0.016 2 0.000 0 0.380 6 0.317 8)	51	7	140	1 030
4	(0.800 8 1.350 0 0.715 2 1.500 0 0.875 0 1.200 0 0.400 0 0.345 0 0.192 0 0 0)	28.564 9	(0.436 4 0.000 0 4.335 9 3.591 5 0.090 0 0.090 0 0.027 5 0.000 0 0.557 7 0.013 3)	(0.526 6 0.000 2 1.615 9 1.865 6 0.076 6 0.072 8 0.016 3 0.000 1 0.379 0 0.288 8)	51	7	150	1 035
5	(0.802 8 1.350 0 0.716 4 1.500 0 0.875 0 1.200 0 0.400 0 0.345 0 0.192 0 0 0)	28.583 3	(0.436 4 0.000 0 4.335 9 3.591 5 0.090 0 0.090 0 0.027 5 0.000 0 0.557 7 0.013 3)	(0.524 5 0.000 0 1.626 1 1.869 8 0.076 5 0.072 9 0.016 2 0.000 3 0.381 3 0.296 3)	51	7	150	1 071
6	(0.801 8 1.350 0 0.714 7 1.500 0 0.875 0 1.200 0 0.400 0 0.345 0 0.192 0 0 0)	28.566 4	(0.436 4 0.000 0 4.335 9 3.591 5 0.090 0 0.090 0 0.027 5 0.000 0 0.557 7 0.013 3)	(0.526 4 0.000 2 1.623 1 1.872 4 0.076 5 0.072 9 0.016 2 0.000 0 0.380 6 0.317 8)	51	7	140	1 030
7	(0.800 8 1.350 0 0.715 2 1.500 0 0.875 0 1.200 0 0.400 0 0.345 0 0.192 0 0 0)	28.564 9	(0.436 4 0.000 0 4.335 9 3.591 5 0.090 0 0.090 0 0.027 5 0.000 0 0.557 7 0.013 3)	(0.526 6 0.000 2 1.615 9 1.865 6 0.076 6 0.072 8 0.016 3 0.000 1 0.379 0 0.288 8)	51	7	150	1 035
8	(0.802 8 1.350 0 0.716 4 1.500 0 0.875 0 1.200 0 0.400 0 0.345 0 0.192 0 0 0)	28.583 3	(0.436 4 0.000 0 4.335 9 3.591 5 0.090 0 0.090 0 0.027 5 0.000 0 0.557 7 0.013 3)	(0.524 5 0.000 0 1.626 1 1.869 8 0.076 5 0.072 9 0.016 2 0.000 3 0.381 3 0.296 3)	51	7	150	1 071
9	(0.801 8 1.350 0 0.714 7 1.500 0 0.875 0 1.200 0 0.400 0 0.345 0 0.192 0 0 0)	28.566 4	(0.436 4 0.000 0 4.335 9 3.591 5 0.090 0 0.090 0 0.027 5 0.000 0 0.557 7 0.013 3)	(0.526 4 0.000 2 1.623 1 1.872 4 0.076 5 0.072 9 0.016 2 0.000 0 0.380 6 0.317 8)	51	7	140	1 030
10	(0.801 8 1.350 0 0.714 7 1.500 0 0.875 0 1.200 0 0.400 0 0.345 0 0.192 0 0 0)	28.566 4	(0.436 4 0.000 0 4.335 9 3.591 5 0.090 0 0.090 0 0.027 5 0.000 0 0.557 7 0.013 3)	(0.526 4 0.000 2 1.623 1 1.872 4 0.076 5 0.072 9 0.016 2 0.000 0 0.380 6 0.317 8)	51	7	140	1 030

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Appendix

$g_i(x)$	Function
$g_1(F_{AL} \le 1\text{ kN})$	$1.16 - 0.371\,7x_2x_4 - 0.009\,31x_2x_{10} - 0.484x_3x_9 + 0.013\,43x_6x_{10}$
$g_2(D_{low} \le 32\text{ mm})$	$46.36 - 9.9x_2 - 12.9x_1x_8 + 0.110\,7x_3x_{10}$
$g_3(D_{mid} \le 32\text{ mm})$	$33.86 + 2.95x_3 + 0.179\,2x_{10} - 5.057x_1x_2 - 11.0x_2x_8 - 0.021\,5x_5x_{10} - 9.98x_7x_8 + 22.0x_8x_9$
$g_4(D_{up} \le 32\text{ mm})$	$28.98 + 3.818x_3 - 4.2x_1x_2 + 0.020\,7x_5x_{10} + 6.63x_6x_9 - 7.7x_7x_8 + 0.32x_9x_{10}$
$g_5(V_{low} \le 32\text{ m/s})$	$0.261 - 0.015\,9x_1x_2 - 0.188x_1x_8 - 0.019x_2x_7 + 0.014\,4x_3x_5 + 0.000\,875\,7x_5x_{10} + 0.080\,45x_6x_9 +$ $0.001\,39x_8x_{11} + 0.000\,015\,75x_{10}x_{11}$
$g_6(V_{mid} \le 32\text{ m/s})$	$0.214 + 0.008\,17x_5 - 0.131x_1x_8 - 0.070\,4x_1x_9 + 0.030\,99x_2x_6 - 0.018x_2x_7 + 0.020\,8x_3x_8 + 0.121x_3x_9 -$ $0.003\,64x_5x_6 + 0.000\,771\,5x_5x_{10} - 0.000\,535\,4x_6x_{10} + 0.001\,21x_8x_{11}$
$g_7(V_{up} \le 32\text{ m/s})$	$0.74 - 0.61x_2 - 0.163x_3x_8 + 0.001\,232x_3x_{10} - 0.166x_7x_9 + 0.227x_2^2$
$g_8(F_{ps} \le 4.01\text{ kN})$	$4.72 - 0.5x_4 - 0.19x_2x_3 - 0.012\,2x_4x_{10} + 0.009\,325x_6x_{10} + 0.000\,191x_{11}^2$
$g_9(v_{B-Pillar} \le 9.9\text{ m/s})$	$10.58 - 0.674x_1x_2 - 1.95x_2x_8 + 0.020\,54x_3x_{10} - 0.019\,8x_4x_{10} + 0.028x_6x_{10}$
$g_{10}(v_{door} \le 15.69\text{ m/s})$	$16.45 - 0.489x_3x_7 - 0.843x_5x_6 + 0.043\,2x_9x_{10} - 0.055\,6x_9x_{11} - 0.000\,786x_{11}^2$

Note: (1) F_{AL} is the dummy abdomen load; D_{up}, D_{mid} and D_{low} are the dummy upper rib, middle rib, and lower rib deflections; V_{up}, V_{mid} and V_{low} are the dummy upper chest, middle chest, and lower chest viscous criterion values, respectively; F_{ps} is the dummy public symphysis force; $v_{B-Pillar}$ is the velocity at the middle B-pillar position; v_{door} is the B-pillar velocity at door belt line.

(2) $\begin{cases} \mu_i^L \le \mu_i \le \mu_i^U,\ i = 1-7, \\ \mu_8,\ \mu_9 = 0.345 \text{ or } 0.192, \text{ enumerated variables denote material properties,} \\ \mu_{10},\ \mu_{11} = 0.0, \text{barrier height and position.} \end{cases}$

(3) All random variables and parameters are assumed normally distributed with standard deviations $\sigma_{1-4,6,7} = 0.03$, $\sigma_5 = 0.05$, $\sigma_{8,9} = 0.006$ and $\sigma_{10,11} = 10$ [4].