Introduction to Propositional Logic

Ling324

Syntax of Propositional Logic

Propositional Logic (PL) is a logical system that is built around the two values TRUE and FALSE, called the TRUTH VALUES.

true = 1; false = 0

1. Any atomic statement (represented with the letters p, q, r, s, ...) is a formula in PL.

   p = It is snowing.
   q = It is cold.
   r = Mary will go to the party.
   s = Jane will go to the party.

2. (a) If ϕ is a formula in PL, then $\neg\phi$ is a formula in PL too.

   "It is not the case that ϕ."

   $\neg p$ = It is not the case that it is snowing = It is not snowing.

   $\neg r$ = It is not the case that Mary will go to the party = Mary will not go to the party.

(b) If ϕ and ψ are formulae in PL, then $(\phi \wedge \psi)$ is a formulae in PL.

“ϕ and ψ.”

$(p \wedge q)$ = It is snowing and it is cold.

(c) If ϕ and ψ are formulae in PL, then $(\phi \vee \psi)$ is a formulae in PL.

“ϕ or ψ.”

$(p \vee q)$ = It is snowing or it is cold.

(d) If ϕ and ψ are formulae in PL, then $(\phi \rightarrow \psi)$ is a formulae in PL.

“if ϕ then ψ.”

$(p \rightarrow q)$ = If it is snowing then it is cold.

(e) If ϕ and ψ are formulae in PL, then $(\phi \leftrightarrow \psi)$ is a formulae in PL.

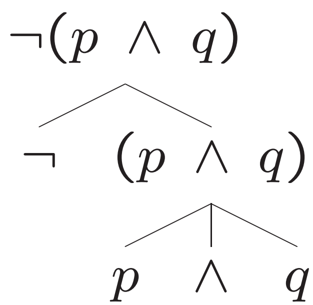
“ϕ if and only if ψ.”

$(r \leftrightarrow s)$ = Mary will go to the party if and only if (iff) Jane will go to the party.

3. Nothing else is a formula in PL.

Syntactic structure

- Syntactic structure of $\neg(p \wedge q)$:



- QUESTION: Construct the syntactic tree for:

  $\neg((p \wedge q) \rightarrow r) \leftrightarrow \neg(s \vee \neg\neg q)$

Semantics of Propositional Logic: Negation

- For any PL formulae ϕ, $[\![\neg\phi]\!] = 1$ iff $[\![\phi]\!] = 0$.

ϕ	$\neg\phi$
1	0
0	1

(1) It is not raining.

Semantics of Propositional Logic: Conjunction

- For any PL formulae ϕ and ψ, $[\![\phi \wedge \psi]\!] = 1$ iff $[\![\phi]\!] = 1$ and $[\![\psi]\!] = 1$.

ϕ	ψ	$\phi \wedge \psi$
1	1	1
1	0	0
0	1	0
0	0	0

(2) Ann is rich and Mason is poor.

- Sometimes, *and* implies that the events expressed in the propositions occurred sequentially.

(3) a. Jenny got married and got pregnant.

b. Jenny got pregnant and got married.

- *But* and *even though* imply contrast between the two propositions and unexpectedness of the connection between the two propositions.

(4) a. Jane snores, but John sleeps well.

b. Even though Jane snores, John sleeps well.

Semantics of Propositional Logic: Conjunction (cont.)

- Truth-conditionally, *and* with sequential meaning, *but*, and *even though* have the same meaning as the propositional logic conjunction.

  The extra meaning of temporal sequence, contrast, or unexpectedness is outside the domain of truth-conditional semantics.

- Sentences with phrasal conjunctions cannot always be directly translated into propositional logic.

(5) a. John and Mary sang.
    b. John smokes and drinks.

(6) a. John and Mary met in New York.
    b. Mary mixed red and blue paint.
    c. At most two children sang and danced.

Semantics of Propositional Logic: Disjunction

- For any PL formulae ϕ and ψ, $[\![\phi \vee \psi]\!] = 1$ iff $[\![\phi]\!] = 1$ or $[\![\psi]\!] = 1$.

ϕ	ψ	$\phi \vee \psi$
1	1	1
1	0	1
0	1	1
0	0	0

(7) It is raining or it is snowing.

- Exclusive *or*

(8) a. All entrees are served with soup or salad.
b. John is in class or at home right now.
c. John was sitting down or standing up.

QUESTION: Let's define semantics for exclusive *or*. $\phi \oplus \psi$ is 1 if either ϕ or ψ is true, but not when both are true.

Semantics of Propositional Logic: Disjunction (cont.)

- Inclusive *or*

  (9) a. At present we invite all passengers who need some extra help, or who are traveling with small children, to board the aircraft.

  b. At the party, Sue did not talk to Fred or Jane.

- Is natural language *or* truth-conditionally ambiguous?

- Sentences with phrasal disjunctions cannot always be directly translated into propositional logic.

  (10) John or Mary will pick you up.

  (11) a. A dentist or a doctor can write prescriptions.

  b. More than half of the students speak Chinese or English.

Conditional

- For any PL formulae ϕ and ψ, $[\![\phi \rightarrow \psi]\!] = 0$ iff $[\![\phi]\!] = 1$ and $[\![\psi]\!] = 0$.

ϕ	ψ	$\phi \rightarrow \psi$
1	1	1
1	0	0
0	1	1
0	0	1

(12) If it snows, it is cold.

- Note that if the antecedent of a conditional statement is false, then the entire conditional statement is true, no matter what the truth value of the consequent of the conditional is.

- Why should this be?

  If the hypothesis is false, we can't be sure whether the conclusion follows from it or not;

  From a false hypothesis, every statement follows.

Semantics of Propositional Logic: Biconditional

- For any PL formulae ϕ and ψ, $[\![\phi \leftrightarrow \psi]\!]$ = 1 iff $[\![\phi]\!]$ = $[\![\psi]\!]$.

ϕ	ψ	$\phi \leftrightarrow \psi$
1	1	1
1	0	0
0	1	0
0	0	1

(13) Janice will go to the party if and only if Joan goes.

Semantics of Propositional Logic (cont.)

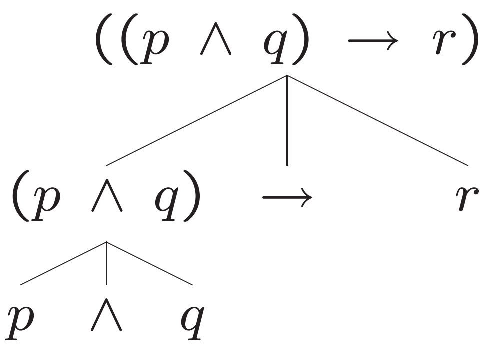
QUESTION: Translate the following sentences into propositional logic formulas.

(14)
a. This engine is noisy and it uses a lot of energy.
b. Joan or Mary left.
c. It is not the case that Cain is guilty and Abel is not.
d. John is not only stupid but also nasty.
e. Johnny wants both a train and a bicycle from Santa Clause, but he will get neither.
f. Charles shows up if Elsa shows up, and the other way around.
g. If father and mother both go, then I won't, but if only father goes, then I will go.

Deriving the Semantic Value of a Complex Formula Compositionally

(15) If Peter and Susan leave, I will be upset.

KEY: p = Peter leaves; q = Susan leaves; r = I will be upset.



p	q	r	$(p \wedge q)$	$((p \wedge q) \rightarrow r)$

QUESTION: Draw the syntactic tree and do the compositional semantic interpretation of the following sentence.

(16) It is not the case that I will be upset if you don't show up.

KEY: p = I will be upset; q = You show up.

Tautologies

- A formula is a TAUTOLOGY iff it is true under any possible assignment of truth values, $[\![\ ]\!]^V$.

- A formula is a TAUTOLOGY iff the final column in its truth table contains nothing but 1's.

  Some examples:

  $(p \vee \neg p)$
  $(p \rightarrow p)$
  $(p \rightarrow (q \rightarrow p))$

Contradictions

- A formula is a CONTRADICTION iff it is false under any $[\![\ ]\!]^V$.

- A formula is a CONTRADICTION iff the final column in its truth table contains nothing but 0's.

  Some examples:

  $\neg(p \vee \neg p)$
  $(p \wedge \neg p)$
  $\neg((p \vee q) \leftrightarrow (q \vee p))$

Contingencies

- A formula is CONTINGENT iff it is true under some $[\![\ ]\!]^V$ and false under some other $[\![\ ]\!]^{V'}$.

- A formula is CONTINGENT iff the final column in its truth table has both 1's and 0's.

  Some examples:

  p
  $(p \vee p)$
  $(p \vee q) \rightarrow q$
  $(p \rightarrow q) \rightarrow p$

Logical Equivalence

- ϕ and ψ are logically equivalent iff for every $[\![\ ]\!]^V$, $[\![\phi]\!]^V = [\![\psi]\!]^V$

  ϕ and ψ are logically equivalent iff $\phi \leftrightarrow \psi$ is a tautology.

  Notation: $\phi \Leftrightarrow \psi$

- Some logical equivalences

Idempotent	$(\phi \vee \phi) \Leftrightarrow \phi$, $(\phi \wedge \phi) \Leftrightarrow \phi$
Commutative	$(\phi \vee \psi) \Leftrightarrow (\psi \vee \phi)$, $(\phi \wedge \psi) \Leftrightarrow (\psi \wedge \phi)$
Complement	$\neg\neg\phi \Leftrightarrow \phi$, $(\phi \vee \neg\phi) \Leftrightarrow T$, $(\phi \wedge \neg\phi) \Leftrightarrow F$
Identity	$(\phi \vee F) \Leftrightarrow \phi$, $(\phi \vee T) \Leftrightarrow T$ $(\phi \wedge F) \Leftrightarrow F$, $(\phi \wedge T) \Leftrightarrow \phi$
Associative	$(\phi \vee (\psi \vee \chi)) \Leftrightarrow (\phi \vee \psi) \vee \chi)$ $(\phi \wedge (\psi \wedge \chi)) \Leftrightarrow (\phi \wedge \psi) \wedge \chi)$
De Morgan's Law	$\neg(\phi \vee \psi) \Leftrightarrow (\neg\phi \wedge \neg\psi)$, $\neg(\phi \wedge \psi) \Leftrightarrow (\neg\phi \vee \neg\psi)$
Contraposition	$(\phi \rightarrow \psi) \Leftrightarrow (\neg\psi \rightarrow \neg\phi)$
Conditional Law	$(\phi \rightarrow \psi) \Leftrightarrow (\neg\phi \vee \psi)$
Biconditional Law	$(\phi \leftrightarrow \psi) \Leftrightarrow ((\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi))$
Distributive Law	$(\phi \vee (\psi \wedge \chi)) \Leftrightarrow ((\phi \vee \psi) \wedge (\phi \vee \chi))$ $(\phi \wedge (\psi \vee \chi)) \Leftrightarrow ((\phi \wedge \psi) \vee (\phi \wedge \chi))$

Logical Equivalence

- Reducing Boolean operations

  Because of these logical equivalences, we can express all Boolean operations in terms of the conjunction and negation operations, or in terms of the disjunction and negation operations.

Entailment

- ϕ entails ψ iff for every $[\![\ ]\!]^V$, if $[\![\phi]\!]^V = 1$ then $[\![\psi]\!]^V = 1$.

- A way to verify whether ϕ entails ψ is to write out a truth table for the corresponding conditional $(\phi \rightarrow \psi)$, and verify that the final column consists entirely of 1's.

  Some examples:

  $(p \wedge q)$ entails p.
  $(p \wedge q)$ entails q.
  p entails $(p \vee q)$.
  q entails $(p \vee q)$.
  $((p \rightarrow q) \wedge p)$ entails q