# MATH 819 – HW2 (SHEAFIFICATION, STRUCTURE SHEAVES, BASICS OF SCHEMES)

**How much to do:** For this assignment, please do (at a minimum) all the instructor problems. You are not required to do every Vakil problem, but you should do several.

**Due date:** In class Thursday, Feb 9th

**Reading:** Vakil Sections 3.1-3.5, 4.1-4.4 (equivalent to Hartshorne Section 2.2 – much more terse!). Next week, 3.6 and 5.1-5.2.

**If you want to read ahead:** Basics on modules (see e.g. Atiyah-Macdonald or Dummit and Foote):

- Definitions: module, homomorphism, submodule, kernel, quotient, localization of a module ($S^{-1}M$ as an $S^{-1}R$-module), cyclic module, finitely-generated module

Instructor problems:

(1) (Reducedness and stalks) Let $R$ be a ring. Show TFAE:
   (i) $R$ is reduced.
   (ii) For all maximal ideals $\mathfrak{m} \subset R$, $R_\mathfrak{m}$ is reduced.
   (iii) For all prime ideals $P \subset R$, $R_P$ is reduced.
   Conclude: for a scheme $X$, the stalk $\mathcal{O}_{X,p}$ is reduced for all $p \in X$ if and only if, for all affine open sets $U \subseteq X$, $\mathcal{O}_X(U)$ is reduced.

(2) (Nilpotents and gluing) Let $(X, \mathcal{O}_X)$ be a ringed space. For each open set $U$, let $\mathcal{N}(U)$ be the nilradical of $\mathcal{O}_X(U)$.
   (a) Show $\mathcal{N}$ is a presheaf.
   (b) Show $\mathcal{N}$ satisfies the identity axiom, but only satisfies gluing for *finite* covers. If $X$ is a Noetherian topological space, use Vakil 3.6.U to show $\mathcal{N}$ is a sheaf.
   (c) Non-Noetherian counterexample to (b) (cf. Vakil 5.2.F):
   Let $X$ be the disjoint union of $X_n = \operatorname{Spec} k[x]/x^n$ for all $n \geq 1$. The global section $(x, x, x, \ldots)$ is locally nilpotent but not globally nilpotent.

(3) (Gluing and generic points) Recall the gluing of $\mathbb{P}^2$ from HW1. In this problem, view the open sets $U_i$ and $U_{ij}$ as affine schemes (i.e. including non-closed points).
   (a) The following are equivalent to one of the gluing maps from HW1:

$$U_2 \supset U_{21} \xleftarrow{\sim} U_{12} \subset U_1$$

$$k[x, y] \hookrightarrow k[x, y, 1/y] \xrightarrow{\phi} k[x, z, 1/z] \hookleftarrow k[x, z]$$

$$x \mapsto x/z,$$

$$y \mapsto 1/z.$$

Let $(f) = (y^2 - x^3 - 1) \in \operatorname{Spec}(k[x,y])$ and $(g) = (z - x^3 - z^3) \in \operatorname{Spec}(k[x,z])$. (You may assume these ideals are prime.) Show that $\phi$ identifies these ideals. *Something similar happens in the third chart, so our gluing maps for* $\mathbb{P}^2$ *also glue together* $\operatorname{Spec} k[x,y]/(f)$ *and* $\operatorname{Spec} k[x,z]/(g)$ *(and one more) to give an elliptic curve* $C \subseteq \mathbb{P}^2$*. Overall, the underlying set of the scheme* $\mathbb{P}^2$ *has just* one *non-closed "generic point" corresponding to* $C$*.*

(b) Which points of $\mathbb{P}^2$ are *not* in $U_2$? (Hint: various closed points but only *one* non-closed point.)

(4) (Pictures of nonreducedness) In this problem, for an ideal $I$ of a ring $R$, by abuse of notation write $\mathbb{V}(I)$ for the scheme $\operatorname{Spec} R/I$. The *scheme-theoretic intersection* $\mathbb{V}(I) \cap \mathbb{V}(J)$ is by definition $\mathbb{V}(I+J) = \operatorname{Spec} R/(I+J)$ (**not** $\operatorname{Spec} R/\sqrt{I+J}$.) We'll examine these concepts further in class.

Let $X = \operatorname{Spec} \dfrac{k[x,y,z]}{(x,z)^2}$. Draw a picture of $X$ (it looks like a tube in $\mathbb{A}^3$).

(a) Let $X' = X \cap \mathbb{V}(x)$. Draw a picture that makes it clear that $X' \subseteq \mathbb{V}(x)$. Does your picture suggest $X'$ is contained in $\mathbb{V}(x+z)$? Check: is $x+z \in (x,z)^2 + (x)$?

(b) Let $X'' = X \cap \mathbb{V}(xy - z)$. Draw a picture of $X''$ in $\mathbb{A}^3$ (it may be helpful to first plot $xy - z = 0$, which is a quadric surface containing the $y$-axis. Try `https://math3d.org`.)
Let $H = \mathbb{V}(ax + bz)$ be *any* plane containing the $y$-axis (assume $a$ and $b$ are not both 0). Show that $X''$ is *not* contained in $H$. That is, the nonreduced structure of $X''$ "twists around", following the shape of $\mathbb{V}(z - xy)$.

(c) Let $Y = \operatorname{Spec} \dfrac{k[x,y]}{xy}$, the union of the $x$ and $y$ axes. Let $Z$ be a tangent vector at the origin pointing in *any* direction (figure out the equations for $Z$). Show that $Z \subseteq Y$. So $Y$ contains the entire 2D "tangent space" at the origin.
This should be surprising – you might have expected $Y$ to only contain the vertical and horizontal tangent vectors. In general for schemes we only have

$$(Y_1 \cup Y_2) \cap Z \supseteq (Y_1 \cap Z) \cup (Y_2 \cap Z).$$

Vakil problems:

- 2.5.C
- 3.2.T (for fun - don't hand in)
- 3.6.ABD (connectedness and irreducibility)
- 3.6.U (noetherian topological spaces)
- 5.1.C - just the claim in the 'hint' (finite union of noetherian topological spaces)