MATH 819 – HW4 (FIBER PRODUCTS, PROJ)

Due date: In class Thursday, March 16th

Reading: Vakil: We previously covered 8.3 and 9.1. Fiber products: 10.1-10.5. Proj: 4.5, 7.4, 16.1. (Please note that I am following the December 2022 version)

(1) Let P be a property of morphisms. We say **P is preserved by pullback** if the following is true: let $\pi : X \to S$ be a morphism with P and let $\alpha : S' \to S$ be any morphism. Then $\pi' : S' \times_S X \to S'$ has P.

Show:

(a) "Locally of finite type" is preserved by pullback. (Hint: this is a local property on both S' and $S' \times_S X$, so reduce to S', S, X and therefore $S' \times_S X$ all affine.)

(b) "Quasicompact morphism" is preserved by pullback. (Hint: this is a local property on S', so reduce to S', S affine. Then show $S' \times_S X$ is covered by finitely-many affines.)

(c) "Affine morphism" is preserved by pullback. (Hint: this is local on S'...)

See Vakil 10.4.D for a longer list.

Solutions. (a) This is local on both the domain and codomain of $\pi' : S' \times_S X \to S'$. So, let $p' \in S' \times_S X'$, let $s' = \pi'(p') \in S'$. It suffices to give an affine neighborhood $U \subset S'$ of s', and an affine neighborhood $V \subset \pi'^{-1}(U)$ of p', where the map is given by a finitely-generated algebra map.

Let x be the image of p' in X and let $s = \alpha(s') = \pi(x)$. First, replace S by any affine neighborhood of s, shrinking X (around x) and S' (around s') accordingly. Then replace X by an affine neighborhood of x and S' by an affine neighborhood of s' (this new S' is our "U"). This is what we've done:

$$\begin{array}{ccc} S' \times_S X & \longrightarrow & X \\ \downarrow & & \downarrow \\ S' & \longrightarrow & S \end{array} \qquad \begin{array}{ccc} p' & \longmapsto & x \\ \downarrow & & \downarrow \\ s' & \longmapsto & s \end{array} \qquad \begin{array}{ccc} V := U \times_W W' & \longrightarrow & W'(\subset \pi^{-1}(W)) \\ \downarrow & & \downarrow \\ U(\subset \alpha^{-1}(W)) & \longrightarrow & W(\subset S) \end{array}$$

We proved in class that this new $S' \times_S X$ is an open subset of the old $S' \times_S X$, and it contains p'. Also, now S', S, X are all affine, so $S' \times_S X$ is affine as well.

We claim that this is our desired neighborhood "V". On the ring map side, the claim is that if $R \to A$ is a finitely-generated R algebra and $R \to R'$ is arbitrary, then $R' \to R' \otimes_R A$ is finitely-generated as an R'-algebra. This is true because $A \cong \frac{R[x_1,\ldots,x_n]}{I}$ for some n and I, and then $R' \otimes_R A \cong \frac{R'[x_1,\ldots,x_n]}{I}$, with the same (finite) generators and relations.

(b) Similar. After reducing we may assume S, S' affine (the details are slightly simpler than in (a) since it doesn't involve X). It's enough to show $S' \times_S X$ is quasicompact, since "quasicompact morphism" is an affine-local property. Since π is quasicompact and S is now affine, X is quasicompact. Then X is covered by finitely-many affines, so $S' \times_S X$ is covered by finitely-many affines, hence is quasicompact.

(c) Even simpler. After reducing we may assume S, S' are affine (again X is not involved). Again it's enough to show $S' \times_S X$ is affine (since "affine morphism" is an affine-local property). Since π is affine, we get X is affine, and then clearly $S' \times_S X$ is affine. □

(2) A morphism $\pi : X \to S$ is **finite** if it is affine and, for each affine open $U = \operatorname{Spec} R \subset S$, $\mathcal{O}_X(\pi^{-1}(U))$ is a finitely-generated R-module. This is an affine-local condition on S (same argument as HW3#7) and is preserved by pullback (in the sense of Problem 2).
   (a) Show that the open embedding $\operatorname{Spec} k[t, t^{-1}] \hookrightarrow \operatorname{Spec} k[t]$ is **not** finite. This shows finiteness is not affine-local on the source (since the identity map $\operatorname{Spec} k[t] \to \operatorname{Spec} k[t]$ is obviously finite.) Show that the map of problem 1(b) **is** finite.
   (b) Let $\pi : X \to S$ be a finite morphism. Show that the fibers of π are finite sets. (Pull back to a fiber $S' = \operatorname{Spec} k(p)$. Then look up and apply this theorem of commutative algebra: an artinian ring has finitely-many prime ideals.)

Solutions. (a) The ring map is $k[t] \to k[t, t^{-1}]$. This isn't finitely generated as a $k[t]$-module because it includes t^{-k} for arbitrarily large k. The map in problem 1(b) was $k[t^2] \to k[t]$. This *is* finite because, as a $k[t^2]$-module, $k[t]$ is generated by 1 and t.

(b) Assume $\pi : X \to S$ is finite. Let $s \in S$ and by abuse of notation, let s also denote the one-point scheme $\operatorname{Spec} k(s)$, which has an inclusion morphism $s \to S$. Since finiteness is preserved by base change, $s \times_S X \to s$ is finite. Since finite morphisms are affine, $s \times_S X$ is affine. That is, the fiber of our morphism is affine and given by a $k(s)$-algebra A that is finite-dimensional as a $k(s)$-*vector space*. In particular, it's obvious that A satisfies the descending chain condition on ideals, hence is an artinian ring. Hence it has only finitely-many prime ideals. □

(3) (Based on NTAG Mar 2) Let $M_n = \mathbb{A}^{n^2}$ be the affine space of $n \times n$ matrices and let $C_n = \{(A, B) : AB = BA\} \subseteq M_n \times M_n$ be the subscheme of commuting matrices. Note that the equations $AB = BA$ define C_n as a *scheme*, and as of 2023 it is not known whether this gives a radical ideal.

Let X be any scheme. Explain why a map $X \to C_n$ (an X-valued point of C_n) is the same as a pair of commuting matrices with entries in $\Gamma(X, \mathcal{O}_X)$. (For example, if $X = \operatorname{Spec} R$, this means a pair of commuting matrices with entries in R.)

Solutions. Since C_n is affine, a map $X \to C_n$ is the same as a ring map of global sections $R \to \Gamma(X, \mathcal{O}_X)$, where R is the ring of global sections of C_n. This is

$$R = \frac{k[a_{ij}, b_{ij} : 1 \leq i, j \leq n]}{(AB = BA)}.$$

Here the equation $AB = BA$ means "the ideal generated by

$$(*) \quad \sum_k a_{ik} b_{kj} - \sum_k b_{ik} a_{kj}$$

for all i, j." A ring map $R \to \Gamma(X, \mathcal{O}_X)$ is given by a choice of where to send the generators,

$$a_{ij} \mapsto f_{ij}, \quad b_{ij} \mapsto g_{ij}, \text{ for some } f_{ij}, g_{ij} \in \Gamma(X, \mathcal{O}_X),$$

such that the equations $(*)$ are satisfied. These equations precisely mean that the matrices $F = (f_{ij})$ and $G = (g_{ij})$ commute. □

(4) (Proj) Let S be an $\mathbb{N}$-graded noetherian ring. Let M be a finitely-generated graded S-module.

(a) Let $f \in S$ be homogeneous of positive degree.
Show: $M_f = 0$ if and only if $f^d M = 0$ for $d \gg 0$.

(b) TFAE: (i) $M_f = 0$ for all homogeneous $f \in S$ of positive degree, (ii) $(S_+)^d M = 0$ for all natural numbers $d \gg 0$, (iii) $M_d = 0$ for $d \gg 0$.

(c) Let $I, J \subseteq S$ be homogeneous ideals and let $U \subset \operatorname{Spec} S$ be the complement of the irrelevant locus. TFAE:
(i) $\operatorname{Proj} S/I = \operatorname{Proj} S/J$, (ii) $(\operatorname{Spec} S/I) \cap U = (\operatorname{Spec} S/J) \cap U$, (iii) $I_d = J_d$ for $d \gg 0$.
(Hint: Compare I to $I \cap J$ and use (b). Show (i) $\Leftrightarrow$ (ii) directly by examining distinguished open sets.)
This shows: two homogeneous ideals I, J define the same projective scheme if and only if their affine schemes agree away from the irrelevant locus.

(d) Give a map $M_0 \to \Gamma(\widetilde{M}, \operatorname{Proj} S)$. Show that this map need not be injective nor surjective.

Solutions.

(a) ($\Rightarrow$) Let M be generated by $m_1, \ldots, m_n$. Since $M_f = 0$, for each i there exists d_i such that $f^{d_i} m_i = 0$. Let d be the max of the d_i. Then $f^d m_i = 0$ for all i, so $f^d M = 0$.

($\Leftarrow$) Since $f^d M = 0$, $f^d m = 0$ for all $m \in M$. Therefore $m/1 = 0$ for all $m \in M$, so $M_f = 0$.

(b) (i) $\Rightarrow$ (ii): Let S_+ be generated by $f_1, \ldots, f_k$. By part (a) and finiteness, we can choose N large enough that $f_i^N M = 0$ for all i. Then $(S_+)^{kN}$ is generated by all products involving kN of the f_i's. By the pigeonhole principle, any such product has a factor f_i^N. Therefore $(S_+)^{kN} M = 0$.

(ii) $\Rightarrow$ (iii) : Suppose $(S_+)^N M = 0$. Let M be generated by $m_1, \dots, m_n$. Let S_+ be generated by $f_1, \dots, f_k$. Let $D = \max \deg(m_j)$ and let $E = \max \deg(f_i)$.

Since M is generated by elements of degree $\leq D$ and S_+ is generated by elements of degree $\leq E$, every element of M of degree $\geq D + NE$ can be expressed as a sum of elements of the form $f \cdot m_j$, where f is a product of at least N of the f_i's. Since $(S_+)^N M = 0$, it follows $f \cdot m_j = 0$. Thus $M_d = 0$ for all $d \geq D + NE$.

(iii) $\Rightarrow$ (i) : Suppose $M_d = 0$ for all $d \gg 0$. Let f be homogeneous of positive degree. Since M is finitely-generated, $f^e M \subseteq \bigoplus_{d' \geq d} M_{d'} = 0$ for $e \gg 0$. Therefore, by part (a), $M_f = 0$.

(c) We need to assume S is generated in degree 1. I don't think the statement is true otherwise. Sorry!

For (i) $\Leftrightarrow$ (ii), the key fact is the following. Let S be a graded ring containing a unit u that is homogeneous of degree 1. Let I, J be homogeneous ideals of S. Then $I = J$ if and only if $I_0 = J_0$. (A similar fact applies to modules.) Proof: if $I = J$, clearly $I_0 = J_0$. Conversely, suppose $I_0 = J_0$. It suffices to show that I and J contain the same homogeneous elements, so let f be homogeneous. Then since u is a unit,

$$f \in I \Leftrightarrow fu^{-\deg(f)} \in I_0 \Leftrightarrow fu^{-\deg(f)} \in J_0 \Leftrightarrow f \in J.$$

Likewise, $S/I = S/J$ if and only if $(S/I)_0 = (S/J)_0$.

In particular, comparing $\operatorname{Proj} R/I$ and $\operatorname{Proj} R/J$ are covered by the sets $D_+(f)$ where f is homogeneous of degree 1. These are Spec of the rings $((R/I)_f)_0$ and $((R/J)_f)_0$. Similarly, $\operatorname{Spec} R/I \cap U$ and $\operatorname{Spec} R/J \cap U$ are covered by the sets $D(f)$, via the rings R/I and R/J. Applying the fact above, we see that the charts of Proj agree if and only if the charts of Spec agree.

For (iii), we apply part (b), equivalence $(iii) \iff (ii)$, to the modules $I/(I \cap J)$ and $J/(I \cap J)$.

(d) Given $m \in M_0$, we get a global section of $\widetilde{M}$ from $\frac{m}{1} \in (M_f)_0$ over all homogeneous f's. This clearly glues on overlaps.

Not injective: Let $M = S/S_+ = S_0$. Then $M_d = 0$ for all $d > 0$, so $\widetilde{M} = 0$ by part (b). But $M_0 = S_0 \neq 0$.

Not surjective: Also by part (b), suppose $M \subset N$ and $M_d = N_d$ for all $d \gg 0$. Then $\widetilde{M} = \widetilde{N}$. So, for example, let $M = S_+$. Then $\widetilde{M} = \widetilde{S} = \mathcal{O}_{\operatorname{Proj} S}$ because $M_d = S_d$ for all $d \geq 1$. So $\Gamma(\widetilde{M}, \operatorname{Proj} S) \neq 0$, but of course $M_0 = (S_+)_0 = 0$.

(5) Consider the graded ring map $\phi^\# : k[X, Y, Z] \to k[S, T]$ defined by $X \mapsto S^3 - ST^2$, $Y \mapsto S^2 T$, $Z \mapsto T^3$.
  (a) Check $\sqrt{\phi^\#((X, Y, Z))} = (S, T)$. Deduce $\phi^\#$ induces a morphism $\phi : \mathbb{P}^1 \to \mathbb{P}^2$ sending $[S : T]$ to $[S^3 - ST^2 : S^2 T : T^3]$.
  (b) As subsets of $\mathbb{P}^1$, what are $\phi^{-1}(\{X = 0\})$, $\phi^{-1}(\{Y = 0\})$, $\phi^{-1}(\{Z = 0\})$? What about as subschemes?

(c) By examining ϕ on the standard affine charts of $\mathbb{P}^2$, show that ϕ is not a closed embedding. (It is enough to check one chart if you think about part (a) carefully.)

Solutions. (a) Since $Z \mapsto T^3$, the radical contains T. Since $X \mapsto S^3 - ST^2$ and the radical contains T, the radical then contains ST^2, hence S^3, hence S.

(b) The equation $X = 0$ corresponds to $S^3 - ST^2 = 0$, or $S(S-T)(S+T) = 0$. This is three reduced points on $\mathbb{P}^1$. Indeed, on the chart $T = 1$, it's just $\operatorname{Spec} \frac{k[s]}{s(s-1)(s+1)}$.

The equation $Y = 0$ corresponds to $S^2T = 0$. This is one reduced point ($[1:0]$) and one double point ($[0:1]$) on $\mathbb{P}^1$. On the two standard charts it's $\frac{k[s]}{s^2}$ and $\frac{k[t]}{t}$.

The equation $Z = 0$ corresponds to $T^3 = 0$. This is one nonreduced point ($[1:0]$) of multiplicity 3 on $\mathbb{P}^1$. On the $S = 1$ chart it's $\frac{k[t]}{t^3}$.

(c) The simplest way to show this is that the morphism isn't one-to-one. On the chart $Z = 1$, the morphism is $\mathbb{A}^1 \to \mathbb{A}^2$ via $s \mapsto (s^3 - s, s^2)$, which sends $\pm 1 \mapsto (0,1)$.

On the ring map side, this chart is $k[x,y] \to k[s]$ by $x \mapsto s^3 - s$ and $y \mapsto s^2$. This isn't a surjective ring map because if we mod out by $(x, y-1)$ (corresponding to the point $(0,1)$ above), we get $\frac{k[x,y]}{(x,y-1)} \to \frac{k[s]}{(s^3-s,s^2-1)} = \frac{k[s]}{(s^2-1)}$, which simplifies to $k \to k^2$, which is no longer surjective. (If $R \to S$ is surjective, so is $R/I \to S/IS$.)

Zhe's proof (thanks Zhe!): Under the ring map $k[x,y] \to k[s]$, x^2 maps to the same thing as $y(y-1)^2$, namely $s^2(s^2-1)^2$. Therefore, given any $f(x,y)$, we can replace powers x^k with $k \geq 2$ by expressions in y without changing its image, and so reduce f to the form

$$f = xg_1(y) + g_2(y).$$

Now when we apply the ring map, the first term gives only odd powers of s and the second term gives only even powers of s. In particular, if $g_1(y) = a_0 + a_1 y + \cdots + a_n y^n$, then

$$\begin{aligned}\phi(xg_1(y)) &= (s^3 - s)(a_0 + a_1 s^2 + \cdots + a_n s^{2n}) \\ &= -a_0 s + (a_0 - a_1)s^3 + \cdots + (a_{n-1} - a_n)s^{2n+1} + a_n s^{2n+3}.\end{aligned}$$

If this equals s, then solving the equations gives $a_0 = -1$, but also $a_0 = a_1 = a_2 = \cdots = a_n = 0$, a contradiction.