MATH 819 – HW4 (FIBER PRODUCTS, PROJ)

Due date: In class Thursday, March 16th

Reading: Vakil: We previously covered 8.3 and 9.1. Fiber products: 10.1-10.5. Proj: 4.5, 7.4, 16.1. (Please note that I am following the December 2022 version)

(1) **Hand in only part (c).** Parts (a)-(b) are essentially shown in Vakil Examples 10.2.3 and 10.3.4, so look there if you get stuck on them.

(a) Show that $\operatorname{Spec}(\mathbb{Q}(\sqrt{2}) \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt{2}))$ is the disjoint union of two points.
This shows that the product of integral schemes over a field need not be integral. However, this does not occur over algebraically closed fields (see Vakil 10.4.H).

(b) Let $T = \operatorname{Spec} \mathbb{Q}[t]$, let $S = \operatorname{Spec} \mathbb{Q}[s]$, and let $\alpha : T \to S$ be the morphism corresponding to the ring map $\mathbb{Q}[s] \to \mathbb{Q}[t]$ given by $s \mapsto t^2$.
Give an example of a fiber of α consisting of (i) two reduced points, (ii) one point $p \in X$, for which the corresponding map of residue fields $k(q) \hookrightarrow k(p)$ is a degree two field extension, (iii) one nonreduced point.

(c) Let $X = \operatorname{Spec} \frac{\mathbb{Q}[s][x,y]}{y^2 - x^3 - s}$, considered as a scheme over $S = \operatorname{Spec} \mathbb{Q}[s]$, a family of (mostly) elliptic curves. Let $\pi : X \to S$ be the corresponding structure morphism.
With $\alpha : T \to S$ from part (b), let $X_T := T \times_S X$ and let $\pi_T : X_T \to T$ be the natural projection (called the **pullback of** π **along** α). Describe the coordinate ring of X_T as a $k[t]$-algebra. Observe that the fibers of X_T at $t = 1$ and $t = -1$ are identical.

(2) Let P be a property of morphisms. We say **P is preserved by pullback** if the following is true: let $\pi : X \to S$ be a morphism with P and let $\alpha : S' \to S$ be any morphism. Then $\pi' : S' \times_S X \to S'$ has P.

Show:

(a) "Locally of finite type" is preserved by pullback. (Hint: this is a local property on both S' and $S' \times_S X$, so reduce to S', S, X and therefore $S' \times_S X$ all affine.)

(b) "Quasicompact morphism" is preserved by pullback. (Hint: this is a local property on S', so reduce to S', S affine. Then show $S' \times_S X$ is covered by finitely-many affines.)

(c) "Affine morphism" is preserved by pullback. (Hint: this is local on S'...)

See Vakil 10.4.D for a longer list.

(3) A morphism $\pi : X \to S$ is **finite** if it is affine and, for each affine open $U = \operatorname{Spec} R \subset S$, $\mathcal{O}_X(\pi^{-1}(U))$ is a finitely-generated R-module. This is an affine-local condition on S (same argument as HW3#7) and is preserved by pullback (in the sense of Problem 2).

  (a) Show that the open embedding $\operatorname{Spec} k[t, t^{-1}] \hookrightarrow \operatorname{Spec} k[t]$ is **not** finite. This shows finiteness is not affine-local on the source (since the identity map $\operatorname{Spec} k[t] \to \operatorname{Spec} k[t]$ is obviously finite.) Show that the map of problem 1(b) **is** finite.

  (b) Let $\pi : X \to S$ be a finite morphism. Show that the fibers of π are finite sets. (Pull back to a fiber $S' = \operatorname{Spec} k(p)$. Then look up and apply this theorem of commutative algebra: an artinian ring has finitely-many prime ideals.)

(4) (Based on NTAG Mar 2) Let $M_n = \mathbb{A}^{n^2}$ be the affine space of $n \times n$ matrices and let $C_n = \{(A, B) : AB = BA\} \subseteq M_n \times M_n$ be the subscheme of commuting matrices. Note that the equations $AB = BA$ define C_n as a *scheme*, and as of 2023 it is not known whether this gives a radical ideal.

Let X be any scheme. Explain why a map $X \to C_n$ (an X-valued point of C_n) is the same as a pair of commuting matrices with entries in $\Gamma(X, \mathcal{O}_X)$. (For example, if $X = \operatorname{Spec} R$, this means a pair of commuting matrices with entries in R.)

(5) (Proj) Let S be an $\mathbb{N}$-graded noetherian ring. Let M be a finitely-generated graded S-module.

  (a) Let $f \in S$ be homogeneous of positive degree.
  Show: $M_f = 0$ if and only if $f^d M = 0$ for $d \gg 0$.

  (b) TFAE: (i) $M_f = 0$ for all homogeneous $f \in S$ of positive degree, (ii) $(S_+)^d M = 0$ for all natural numbers $d \gg 0$, (iii) $M_d = 0$ for $d \gg 0$.

  (c) Let $I, J \subseteq S$ be homogeneous ideals and let $U \subset \operatorname{Spec} S$ be the complement of the irrelevant locus. TFAE:
  (i) $\operatorname{Proj} S/I = \operatorname{Proj} S/J$, (ii) $(\operatorname{Spec} S/I) \cap U = (\operatorname{Spec} S/J) \cap U$, (iii) $I_d = J_d$ for $d \gg 0$.
  (Hint: Compare I to $I \cap J$ and use (b). Show (i) $\Leftrightarrow$ (ii) directly by examining distinguished open sets.)
  This shows: two homogeneous ideals I, J define the same projective scheme if and only if their affine schemes agree away from the irrelevant locus.

  (d) Give a map $M_0 \to \Gamma(\widetilde{M}, \operatorname{Proj} S)$. Show that this map need not be injective nor surjective.

(6) Consider the graded ring map $\phi^\# : k[X, Y, Z] \to k[S, T]$ defined by $X \mapsto S^3 - ST^2$, $Y \mapsto S^2 T$, $Z \mapsto T^3$.

  (a) Check $\sqrt{\phi^\#((X, Y, Z))} = (S, T)$. Deduce $\phi^\#$ induces a morphism $\phi : \mathbb{P}^1 \to \mathbb{P}^2$ sending $[S : T]$ to $[S^3 - ST^2 : S^2 T : T^3]$.

  (b) As subsets of $\mathbb{P}^1$, what are $\phi^{-1}(\{X = 0\})$, $\phi^{-1}(\{Y = 0\})$, $\phi^{-1}(\{Z = 0\})$? What about as subschemes?

(c) By examining ϕ on the standard affine charts of $\mathbb{P}^2$, show that ϕ is not a closed embedding. (It is enough to check one chart if you think about part (a) carefully.)