MATH 819 – HW5 (SEPARATED AND PROPER; LOCALLY FREE SHEAVES)

Due date: In class Tuesday, April 4th

Reading: Vakil: Separated and proper (Chapter 11) – see also Hartshorne II.3. Vector bundles and locally free sheaves: Chapter 14.1-14.3 and 15. Global Spec and Global Proj: 18.1-18.2

(1) (Separatedness)
  (a) If $\pi : X \to S$ is separated, $S = \mathrm{Spec}(R)$ is affine, and $U, V \in X$ are affine open sets, then $U \cap V$ is affine. (Examine $\Delta^{-1}(U \times_S V)$.)
  Taking $X = S$ and $\pi = \mathrm{id}$, conclude: in an affine scheme, intersections of arbitrary affine open subschemes are affine. Then show $\mathbb{A}^2$ with two origins (over any base scheme) is not affine.
  (b) Show that separatedness is preserved by pullback, and properness is preserved by composition. (Use the definition, *or* assume X, S are locally noetherian and π is finite type and then use the valuative criterion).

(2) Let R be a noetherian ring and M a finitely-generated R-module.
  (a) Let S be a multiplicative set and suppose $S^{-1}M = 0$. Show that there exists $f \in S$ such that $M_f = 0$.
  (b) Let $p \in \mathrm{Spec}(R)$ and suppose M_p is a free R_p-module of rank n. Show that there exists $f \in R - p$ such that M_f is a free R_f-module of rank n.
  (Hint: First find a map $R^n \to M$ such that $R_p^n \to M_p$ is an isomorphism. Then use (a).)
  (c) Let X be a noetherian scheme and $\mathscr{F}$ a coherent sheaf on X. Show: $\mathscr{F}$ is locally free of rank $n \Leftrightarrow$ for all $p \in X$, $\mathscr{F}_p$ is a free $\mathcal{O}_{X,p}$-module of rank n.

(3) Let $\mathscr{E}, \mathscr{F}$ be locally free sheaves of ranks e and f on a noetherian scheme X.
  (a) Show that $\mathscr{E} \otimes \mathscr{F}$, $\mathscr{E} \oplus \mathscr{F}$ and $\mathscr{H}om(\mathscr{E}, \mathscr{F})$ are all locally free, of ranks ef, $e+f$ and ef again. (Take for granted that $\mathscr{H}om(\mathscr{E}, \mathscr{F})$ is quasicoherent and given by $\mathrm{Hom}_R(E, F)$ on affine charts, when $\mathscr{F}$ is quasicoherent and $\mathscr{E}$ is coherent; see Vakil 1.6.G and 14.3.A.)
  (b) Show that $\mathrm{Sym}^d(\mathscr{E})$ is locally free of rank $\binom{e+d-1}{d}$. (If $m_1, \ldots, m_e$ is a basis for $\mathscr{E}$ locally, the monomials in $m_1, \ldots, m_e$ give a basis for $\mathrm{Sym}^d(\mathscr{E})$.)
  (c) Show that there is a surjective map of sheaves $\mathscr{E} \otimes \mathscr{H}om(\mathscr{E}, \mathscr{F}) \to \mathscr{F}$ given by "evaluation" $v \otimes \varphi \mapsto \varphi(v)$. (Show it exists via universal properties of $\otimes$ and gluing, then check it is surjective on sufficiently small affine charts.)

(d) Specialize part (b) to $\mathscr{F} = \mathcal{O}_X$; the sheaf $\mathscr{H}om(\mathscr{E}, \mathcal{O}_X)$ is called the *dual* of $\mathscr{E}$ and denoted $\mathscr{E}^*$ or $\mathscr{E}^\vee$.
Show: If $\mathscr{E}$ has rank 1, the map $\mathscr{E} \otimes \mathscr{E}^* \to \mathcal{O}_X$ is actually an isomorphism.

(4) Let $R = k[S^4, ST^3, S^3T, T^4] \subset k[S,T]$ (i.e. the 4-th Veronese subring omitting S^2T^2). We reset the grading on S and count its four generators now as degree 1. You may take for granted that $S \cong \frac{k[X,Y,Z,W]}{(XW - YZ, Y^3 - ZW^2, Z^3 - X^2Y)}$.
(a) Verify that $\sqrt{(X,W)} = R_+$, so $D_+(X) \cup D_+(W) = \operatorname{Proj} R$.
(b) Show that $D_+(X)$ and $D_+(W)$ are each isomorphic to $\mathbb{A}^1$ and that the gluing map simplifies to the usual $\mathbb{P}^1$ gluing map, so $\operatorname{Proj} R \cong \mathbb{P}^1$.
(c) Examine $\widetilde{R(1)}$ on each of the charts $D_+(X)$ and $D_+(W)$: write down the generator and transition map. Recognize $\widetilde{R(1)}$ as what we would have called $\mathcal{O}(4)$ on $\operatorname{Proj} k[S,T]$. In particular, its global sections are

$$\Gamma(\operatorname{Proj} R, \widetilde{R(1)}) = k \cdot \{S^4, S^3T, S^2T^2, ST^3, T^4\},$$

even though $S^2T^2 \notin R$. This gives another example where the map $M_0 \to \Gamma(\widetilde{M}, \operatorname{Proj} R)$ isn't surjective. (It may be helpful to write S^2T^2 in terms of X, Y, Z, W on each chart.)

(5) (A valuative criterion) For any ring R and for $d \le n$, let:
- $\mathrm{Mat}_{d\times n}(R)$ be the set of $d \times n$ matrices M with entries in R,
- $U_{d\times n}(R) \subset \mathrm{Mat}_{d\times n}(R)$ be the set of M such that the $d \times d$ minors of M generate the unit ideal in R, called *full rank* matrices.
- $GL_d(R) := U_{d,d}(R)$, the square matrices M such that $\det(M)$ is a unit.

One indirect "definition" of the *Grassmannian* $Gr(d,n)$ is to define, for all *affine* schemes $X = \mathrm{Spec}(R)$,

$$(*) \qquad \mathrm{Hom}(\operatorname{Spec} R, Gr(d,n)) := U_{d\times n}(R)/\sim,$$

where $M \sim AM$ for all $A \in GL_d(R)$. That is, by definition, a map $\operatorname{Spec} R \to Gr(d,n)$ "is" a full-rank $d \times n$ matrix over R, up to the equivalence relation of row operations.

For this problem, ignore the question of how $Gr(d,n)$ is a scheme and just work directly with the definition $(*)$ above. (This is essentially a definition via universal property of $Gr(d,n)$ as a quotient space.)
(a) Let k be a field. Show that $\mathrm{Hom}(\operatorname{Spec} k, Gr(d,n))$ — the k-points of $Gr(d,n)$ — is in bijection with the set of all d-dimensional subspaces $V \subset k^n$.
(b) Let $K = k(t)$, the field of rational functions, with valuation $\mathrm{val}(f)$ given by the order of vanishing of f at $t = 0$. Consider the matrix:

$$M = \begin{bmatrix} 1-t & 1-t^2 & t & t^2 \\ \frac{1}{t^2} & \frac{1+t}{t^2} & 1 & t \end{bmatrix} \in \mathrm{Mat}_{2\times 4}(k(t))$$

Check:

  - Some minor of M is nonzero, so M represents a "morphism $\operatorname{Spec} k(t) \to Gr(2,4)$". For generic t, we have a 2-dimensional subspace of k^4.
  - If we set $t = 0$, the matrix is undefined. If we try rescaling the second row by t^2 and *then* set $t = 0$, the resulting matrix not full-rank over k. So it may seem that we can't "take the limit as $t \to 0$".

(c) Calculate the valuation of each minor of M. You should find two minors are identically zero (order $+\infty$), two are order -1 and two are order 0.

(d) Calculate $A^{-1}M$, where A is columns 1 and 3 of M. You should find that all nonzero entries now have nonnegative valuation, i.e., $A^{-1}M \in \mathrm{Mat}_{2\times 4}(k[t]_{(t)})$. *Now* set $t = 0$ and describe the resulting two-dimensional subspace of k^4.
(Really what has happened is all *minors* now have nonnegative valuation. Since $A^{-1}M$ contains an identity matrix, this includes all individual *entries*.)

(e) Explain why every morphism $\operatorname{Spec} k(t) \to Gr(2,4)$ extends to a morphism $\operatorname{Spec} k[t]_{(t)} \to Gr(2,4)$. This is the "existence" part of the valuative criterion and is one way to prove that $Gr(2,4)$ is proper.