1. Solution: (a) The position is given by the permutation $(1\ 3\ 12\ 11\ 8\ 13\ 9\ 16\ 7\ 6)(2\ 10\ 14\ 4)$ which is even, but the empty space is in box 7 which is an odd parity box (i.e. it is an odd number of moves away from box 16). Therefore the puzzle is not solvable.

   (b) The position is given by the permutation $(1\ 2\ 3\ 4\ 8\ 12\ 16)(10\ 14\ 13)$ which is even, and the empty space is in box 1 which is an even parity box. Therefore the puzzle is solvable.

   A sequence of moves that solves the puzzle is:

$$\ell\ell\ell urrru(\ell urd)d\ell\ell\ell uu$$

   The four moves in parentheses does the 3-cycle of tiles 10, 13, 14.

2. Solution: An element of order 10 must contain a 2-cycle and a 5-cycle (since there are only eight numbers to make cycles with we can't have a 10-cycle). To be even it must actually contain at least two 2-cycles and a 5-cycle, but this requires at least 9 numbers. Therefore, A_8 does not contain an element of order 10.

3. Solution: Consider $\alpha = (1\ 2)(3\ 4)$, $\beta = (3\ 5)$. Both have order 2, and $\alpha\beta = (1\ 2)(3\ 4\ 5)$ has order 6.

4. Solution: One could exhaustively go through all powers of α and determine which ones are 15-cycles, however we could do this more simply by considering the order of a permutation. First note that α has order 15, so α raised to a power k which contains a divisor of 15 will partially "kill" α (i.e. α^k will have order smaller than 15). For example α^3 has order 5 and can only consist of 5-cycles. As another example, α^{10} has order 3 and must consist of 3 cycles. Therefore, we have the following:

$$\begin{aligned} \alpha^k \text{ is a } 15\text{-cycle} \quad &\Leftrightarrow \quad \gcd(k, 15) = 1 \\ &\Leftrightarrow \quad k = 1, 2, 4, 7, 8, 11, 13, 14 \end{aligned}$$

5. Solution: $\beta = (1\ 3\ 5)(2\ 4\ 7\ 6\ 8) = (1\ 3\ 5)(2\ 4\ 7)(2\ 6\ 8)$. A solution consists of a product of 3-cycles, say γ, for which $\beta\gamma = \epsilon$. Therefore, $\gamma = \beta^{-1} = (2\ 8\ 6)(2\ 7\ 4)(1\ 5\ 3)$, and the sequence of moves is

$$(2\ 8\ 6), (2\ 7\ 4), (1\ 5\ 3).$$

6. Solution: If $\alpha \in S_n$ is a non-identity solution to $\alpha^5 = \epsilon$ then α has order 5. Therefore, $\{\alpha, \alpha^2, \alpha^3, \alpha^4\}$ is a collection of 4 distinct solutions to the equation (since each has order 5 too). Therefore, non-identity solutions occur in quadruplets. Since the identity is also a solution the number of solutions is $4k+1$.

7. Solution: To find the maximum order of an element in S_{10} we need to find a product of product of cycles lengths that have the largest least common multiple. For example, there is a permutation of order 21 since a product of a 3-cycle and a 7-cycle has this order. Are there permutations with larger order? We need to find integers $k_1, \ldots, k_m$ such that $k_1 + \cdots + k_m \leq 10$ and $\operatorname{lcm}(k_1, \ldots, k_m)$ is as large as possible. After a bit of trial and error, we find the following.

   $(1\ 2\ 3\ 4\ 5)(6\ 7\ 8)(9\ 10)$ has order 30, and this is the maximum order in S_{10}.

   This is product of a 5-cycle, 3-cycle, and a 2-cycle, therefore it is an odd permutation, so not in A_{10}. the maximum order of an element in A_{10} is 21.

   $(1\ 2\ 3\ 4\ 5\ 6\ 7)(8\ 9\ 10)$ has order 21, and this is the maximum order in A_{10}.

8. Solution: To have order 60 it must also contain a 5-cycle. Also, to be even it must contain another cycle of even length since the 4-cycle is an odd permutation. For example,

$$(1\ 2\ 3)(4\ 5\ 6\ 7)(8\ 9\ 10\ 11\ 12)(13\ 14)$$

is an even permutation, contains the necessary cycles, and has order $\mathrm{lcm}(3, 4, 5, 2) = \mathrm{lcm}(3, 4, 5) = 60$.

9. Solution: Order 3 elements are either 3-cycles or a product of two 3-cycles.

$(\_\ \_\ \_)$: there are $\binom{7}{3}$ choices for the numbers to use in the cycle, and 2 different cycles for each choice of 3 numbers. Therefore, there are $2\binom{7}{3} = 2\dfrac{7 \cdot 6 \cdot 5 \cdot 4}{4 \cdot 3 \cdot 2} = 70$ single 3-cycles.

$(\_\ \_\ \_)(\_\ \_\ \_)$: there are $\binom{7}{3}$ choices for the numbers to use in the first cycle and $\binom{4}{3}$ choices for numbers in the second. There are 2 different cycles that can be made from each choice of 3 numbers, but the order in which the 3-cycles are multiplied doesn't matter. Therefore, there are $4\binom{7}{3}\binom{4}{3}/2 = 8\dfrac{7 \cdot 6 \cdot 5 \cdot 4}{4 \cdot 3 \cdot 2} = 280$ products of two 3-cycles.

Therefore, there are 350 elements of order 3.

10. Solution:

1	2	3	4	5	6
3	2	1	4	5	6

11. Solution: **(a)** $\beta = (1\ 6\ 4\ 7)(2\ 3\ 5)$ **(b)** $\alpha = (4\ 6\ 7)$
(c) $\gamma = (1\ 7)(2\ 3\ 5)$ and $\beta\alpha = (1\ 6\ 4\ 7)(2\ 3\ 5)(4\ 6\ 7) = (1\ 7)(2\ 3\ 5) = \gamma$.

12. Solution: For any $1 \leq i < j \leq n$, let $j = i + m$, then we have

$$(i\ j) = (i\ i{+}m) = (i\ i{+}1)(i{+}1\ i{+}2) \cdots (i{+}m{-}2\ i{+}m{-}1)(i + m - 1\ i + m)(i{+}m{-}1\ i{+}m{-}2) \cdots (i{+}1\ i{+}2)(i\ i{+}1)$$

This has the form $\gamma(i + m - 1\ i + m)\gamma^{-1}$, where γ moves tile i to the right by swapping with its neighbour each step.

In other words, to swap i and j first move i to the right by swapping with its neighbour each time, then once it is next to j swap i and j. Then move j to the left by swapping with its neighbour each time, until it is in box i.

13. Solution: For any $2 \leq a, b \leq n$ we have

$$(a\ b) = (1\ a)(1\ b)(1\ a).$$

Therefore, every 2-cycle is obtainable as a product of 2-cycles of the form $(1\ m)$.

Since every permutation in S_n can be written as a product of 2-cycles, and we've just whown each 2-cycle is a product of ones of the form $(1\ m)$, the result follows.

14. Solution: $(\Longrightarrow)$ Each move $(a\ b)(c\ d)$ is even, and so any product of moves is even. Therefore only even permutations are possible.

$(\Longleftarrow)$ To show every even permutation is possible we'll show we can obtain any 3-cycle. Lets create the 3-cycle $(a\ b\ c)$. Since $n \geq 5$ there are two other boxes, say d and e. Now,

$$\begin{aligned}(a\ b\ c) &= (a\ b)(a\ c)\\ &= (a\ b)(d\ e)(d\ e)(a\ c)\end{aligned}$$

which is a product of two legal moves: $(a\ b)(d\ e)$ and $(d\ e)(a\ c)$.

Therefore, any 3-cycle is possible, hence any even permutation is possible.