Math 5320

Homework 11

From the book, do Ch. 16: 2.2, 4.1, 6.1, 6.2, 6.3 and the problems below. They will help with some of the book problems.

*(1) Prove the following generalization of Proposition 15.2.8: Let F be a field and let $\alpha_1, \alpha_2 \in K/F$ and $\beta_1, \beta_2 \in L/F$. There is an F-isomorphism $\sigma : F(\alpha_1, \alpha_2) \to F(\beta_1, \beta_2)$ that sends α_i to β_i if and only if the irreducible polynomials of α_i and β_i are equal. How would you generalize the statement of the theorem and your proof to say something about F-isomorphisms of $F(\alpha_1, \ldots, \alpha_n)$ and $F(\beta_1, \ldots, \beta_n)$?

Given a monic polynomial $f(x) = x^n - a_1 x^{n-1} + \cdots \pm a_n \in F[x]$, with roots $\alpha_1, \ldots, \alpha_n$, its *discriminant* is defined to be the product of the squares of the differences of every pair of roots, that is, the following product:

$$\prod_{i<j} (\alpha_i - \alpha_j)^2$$

The discriminant of $f(x)$ is 0 if and only if $f(x)$ has multiple roots.

Since the coefficients a_i of $f(x)$ are elementary symmetric functions in the α_i and the discriminant is a symmetric function in the α_i, it is possible to find general formulas for the discriminant in terms of the coefficients (though these can get quite messy when $f(x)$ has a high degree. If $f(x)$ is cubic, then its discriminant is $-4a_1^3 a_3 + a_1^2 a_2^2 + 18 a_1 a_2 a_3 - 4a_2^3 - 27 a_3^2$.

(2) Consider a quadratic polynomial $x^2 + bx + c$. Find its roots using the quadratic formula and then use those to compute the discriminant as defined above. Does this discriminant coincide with the term inside the square root in the quadratic formula?

(3) Explain why the square root of the discriminant of a polynomial $f(x)$ is always contained in its splitting field.

* This assertion is false. For example it would imply $\mathbb{Q}(\sqrt[3]{2}, \zeta_3 \sqrt[3]{2})$ is isomorphic to $\mathbb{Q}(\sqrt[3]{2}, \sqrt[3]{2}) = \mathbb{Q}(\sqrt[3]{2})$, which is not the case. The statement can be fixed by also asserting that the irreducible polynomials of α_2 and β_2 over $F(\alpha_1)$ and $F(\alpha_2)$ are isomorphic via the isomorphism $F(\alpha_1) \to F(\alpha_2)$ that sends α_1 to α_2.

This is related to the problem from the textbook asking us to find all of the $\mathbb{Q}$-automorphisms of $\mathbb{Q}(\sqrt[3]{2}, \zeta_3)$. We know that any such autorphism must send $\sqrt[3]{2}$ to another root of $x^3 - 2$ and ζ_3 to another root of $x^2 + x + 1$, but we need to make sure that any such choice actually defines an automorphism. However, ζ_3 and ζ_3^2 both have the irreducible polynomial over $\mathbb{Q}(\gamma)$ where γ is any choice of root of $x^3 - 2$ and so we may use our corrected version of problem (1).