Math 5320, 4/9/18 Name:

Worksheet 31:

Theorem 16.6.6 uses the terminology of a "group operation on a set" also called "group actions" which doesn't seem to be defined in the book. We've already seen somethings where these ideas are used implicitly, but we'll give some definitions here.

Let G be a group and S be a set. A (left) group operation of G on S is a function

$$\varphi : G \times S \to S : (g, s) \mapsto \varphi(g, s)$$

such that $\varphi(e, s) = s$ for all $s \in S$ and for any $g, h \in G$, $\varphi(gh, s) = \varphi(g, \varphi(h, s))$. We usually leave off the "φ" notation and just write gs for $\varphi(g, s)$, so our two axioms are written simply as $es = s$ and $(gh)s = g(hs)$.

A group action is *faithful* if for any choice of distinct $g, h \in G$, there exists an $s \in S$ such that $gs \neq hs$.

A group action is *transitive* if for any $s, t \in S$, there exists some $g \in G$ such that $gs = t$.

1. We've already seen a group action: the symmetric group S_n acting on the set of polynomials in n variables is a group action. Is this action faithful? Is it transitive? Do the answers change if we restrict to symmetric polynomials or polynomials of a fixed degree?

2. Let K/F be a Galois extension with Galois group G and let $g(x) \in F[x]$ be a polynomial that splits completely in K. Let its roots be $\{\beta_1, \cdots, \beta_r\}$.

   There is a group action of G on this set of roots.

   (a) Is the action of $G(\mathbb{Q}(\zeta_8)/\mathbb{Q})$ on the roots of $x^2 + 1$ faithful? (Try letting g and h be the identity automorphism and the autmorphism sending ζ_8 to ζ_8^3, respectively.

   (b) Is the action of $G(\mathbb{Q}(\sqrt{2}, \sqrt{3})/\mathbb{Q})$ on the roots of $(x^2 - 2)(x^2 - 3)$ transitive?

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