1 The Transylvanian lottery

Suppose you're playing the lottery. Three numbers from between 1 and 7 will be drawn and tickets have three numbers on them. However, you'll win something if you have a ticket with two out of the three numbers.

1. How many possible different lottery tickets could you buy?
2. Suppose you want to buy a bunch of tickets, and you want to buy enough to make sure that you win. What's the minimum number of tickets you can buy to make sure that you'll have at least one ticket with at least two out of three of the lottery numbers on it?

2 The Fano plane

1. Consider the numbers $0, 1, 2, 3, 4$ under addition *and* multiplication mod 5. Check that this satisfies the field axioms. We call this field $\mathbb{F}_5$.
2. Let p be any prime. Consider the numbers $0, 1, 2, \ldots, p-1$ under addition *and* multiplication mod p. Will this satisfy the field axioms? Does the answer depend at all on your choice of p?
3. Consider the projective plane with coordinates in $\mathbb{F}_2$. This happens to have a special name and it's called the Fano plane. How many points does it have?
4. Lines in the projective plane are all equations the form $ax + by + cz = 0$. If we're looking at the coordinates in $\mathbb{F}_2$, the coefficients will also be $\mathbb{F}_2$. What are all the possible lines with coordinates in $\mathbb{F}_2$? Which of the points in the previous problem are on each of these lines? How many points total are on each of the lines?
5. Looking at your answer to the previous problem, how many lines is each of the points on?
6. Check that the Fano plane satisfies the two following conditions:
   (a) Every pair of points is contained in exactly one common line.
   (b) Every pair of lines contains exactly one common point.

   The Fano plane is the smallest (by which I mean least number of total points) set of points and lines that satisfies these rules.
7. Since we're working mod 2 and in projective space, it's not so clear how to plot the Fano plane. However, it can be helpful to draw it in a schematic kind of way: draw 7 points, label them, and then draw each of the lines you found by drawing them passing through each of the points on each one (you won't be able to draw them so that they're all straight).
8. The Fano plane actually has to do with the lottery problem from earlier! Label the points on your drawing as $1, 2, 3, 4, 5, 6, 7$. Write down the sets of three numbers that correspond to each of the lines in the Fano plane. What you've just written down is actually one possible choice of lottery tickets so that you'll have at least one ticket with at least two out of three of the lottery numbers on it, and this is actually the lowest number of tickets that you could buy. How do these statements relate to problem 6?