1 Continued Fractions

Generally, continued fractions are of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}$$

Sometimes, to save space, we write them just as the sequence of numbers $a_0, a_1, a_2, a_3, \ldots$.

1. Compute the following:

   (a) $\cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{2}}}$

   (b) $2 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{4}}}}$

2. Note that continued fraction expressions aren't unique. Verify that the following are both equal to $\frac{3}{4}$:

$$\cfrac{1}{1 + \cfrac{1}{3}} \qquad \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1}}}$$

   Can you find another continued fraction expressions for $\frac{3}{4}$?

3. The Euclidean algorithm provides one way of finding a continued expansion of a given rational number.

   (a) Perform the Euclidean algorithm on 67 and 24 (specifically, start by dividing 67 by 24), writing your steps neatly since you'll want to refer back to it.

   (b) What are the quotients you get in each successive step of the Euclidean algorithm? Compare them to the quotients in the continued fraction in 1(b).

   (c) The following steps show how the first few steps of the Euclidean algorithm can be used to help find the continued fraction for $\frac{67}{24}$. Finish the process to find the continued fraction:

$$\frac{67}{24} = 2 + \frac{19}{24} = 2 + \frac{1}{\frac{19}{24}} = 2 + \frac{1}{1 + \frac{5}{19}} = 2 + \cfrac{1}{1 + \cfrac{1}{\frac{5}{19}}} = \cdots$$

4. Use the Euclidean algorithm to find continued fraction expansions of $\frac{91}{16}$ and $\frac{81}{25}$. After you find the continued fractions, check that when you simplify them, you get the numbers we started with.

5. Some continued fractions aren't finite. Consider the continued fraction

$$1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}}}$$

that is, the continued fraction corresponding to $1, 2, 2, 2, 2, 2, \ldots$.

   (a) What do you think it might be? What do the continued fractions corresponding to the sequences $1, 2$ and $1, 2, 2$ and $1, 2, 2, 2$ sum to?

   (b) There's a very helpful method for solving these types of problems. Set

$$x = \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}}}$$

   Note that we can write the continued fraction from the last problem as $1 + x$ or $1 + \frac{1}{2+x}$. Set these two expressions equal to one another and solve for x. What is the continued fraction from part (a) equal to?

6. What is the continued fraction corresponding to the sequence $1, 3, 1, 3, 1, 3, \ldots$? What is the continued fraction corresponding to the sequence $1, 3, 5, 1, 3, 5, 1, 3, 5, \ldots$?

7. Consider the following statement: A number is rational if and only if we can write it as a continued fraction that terminates. Does this seem reasonable?

8. In lecture, we will take a look at how to approximate irrational numbers with continued fractions. Use the method to find fractions approximating some of your favorite irrational numbers, like π, $\sqrt{2}$, $\sqrt{3}$, etc.

2 The Farey diagram

The Farey diagram is a diagram that's usually drawn in the hyperbolic plane.

We'll draw it in the Poincaré disk model, so start with a circle. Label the point furthest to the right with the fraction $0/1$ and the point furthest to the left with the fraction $1/0$. Connect the two points. Label north pole with the fraction $1/1$ and the south pole with the fraction $-1/1$ and draw (hyperbolic!) lines connecting each of them with the first two points you draw. The Farey diagram is then constructed by picking any two points already connected by a line, and connecting each of those to the point halfway between them. Note that each of these steps creates a hyperbolic triangle. If the labels at the two ends of the long edge of a triangle are a/b and c/d, the label on the third vertex of the triangle is $\frac{a+c}{b+d}$. This process could go on infinitely, so it's ok to stop afer a few steps after you start to get the feel for how the drawing goes. There's a copy of a Farey diagram at the end of the worksheet to help give a sense of what this should look like.

1. Some things to notice about the Farey diagram:

   (a) None of the lines we draw ever cross, so this is a triangular tiling of the Poincaré disk by triangles.

   (b) The labels we make are always in lowest terms. We can check this inductively:

      i. The base case: Consider the first few labels we made, $1/1$, $0/1$, $1/0$, $-1/1$. Check that for all of these that if a/b and c/d have a line connecting them, then $ad - bc = \pm 1$.

ii. Consider two points that have a line connecting them on the Farey diagram. Call their labels a/b and c/d. The point halfway between them is labelled $\frac{a+c}{b+d}$. Show that $a \cdot d - b \cdot c = a \cdot (b+d) - b \cdot (a+c)$.

iii. Why do the last two steps show that if a/b and c/d are both in lowest terms (i.e. the numerator and denominator have a gcd of 1), then so is $\frac{a+c}{b+d}$.

2. Although our students might wish that $\frac{a+c}{b+d}$ is equal to the sum of the fractions $\frac{a}{b}$ and $\frac{c}{d}$, it's not. However, one interesting property that $\frac{a+c}{b+d}$ has is that it's always in between $\frac{a}{b}$ and $\frac{c}{d}$. Show it's true, though to make the problem more reasonable, just in the case where all the numbers are positive. To get started, it may be helpful to check it for some examples.

3. The continued fraction for 3/8 corresponds to the list $0, 2, 1, 2$. Simplify the continued fractions corre-ponding to the following lists of numbers:

   (a) 0

   (b) $0, 2$

   (c) $0, 2, 1$

   (d) $0, 2, 1, 2$

   They are called the *convergents* of 3/8. Check that there is a finite sequence of lines connecting each of the convergents on the Farey diagram. What's the minimum number of lines connecting each successive pair of convergents?

4. Find the continued fraction for 7/16, find its convergents, and check that you can connect them on the Farey diagram as in the last problem.

5. Suppose we have a continued fraction corresponding to $a_0, a_1, a_2, \ldots$. Form its convergents. Label them $\frac{p_0}{q_0}, \frac{p_1}{q_1}, \frac{p_2}{q_2}, \ldots$. Show that for any n, the following expression holds:

$$\frac{p_{n+2}}{q_{n+2}} = \frac{a_{n+2}p_{n+1} + p_n}{a_{n+2}q_{n+1} + q_n}$$

   How does this observation shine light on what you saw in the last two problems?

6. Since every rational number can be written as a continued fraction that terminates, the work we've done in the last few problems actually means that every rational number appears as a label on the Farey diagram!

7. Consider the fraction 3/8 again. Note that any other point a/b that it shares an edge with is a solution to $3b - 8a = \pm 1$. What's going on here?