More Groups

1 More about cosets

1. Pick one corner of a square. What are all the symmetries of the square that send that corner to itself? How many of them are there? Why do they form a subgroup of the group of symmetries of a square?

2. Describe all of the cosets of the subgroup from the first question in the symmetry group of the square. How many of them are there?

3. In cases like the last problem, sometimes it's possible to describe all the cosets of a particular group in a reasonable way, and this helps us count. Use our discussion of cosets of a subgroup partitioning the group they're inside and your answers to the first two problems to find the total number of the symmetries of the square again.

4. What are the cosets of each of these subgroups of the cube from the last problem? How many of them are there?

5. Use the numbers of cosets from the last problem to find the size of the symmetry group of the cube two different ways, verifying what you found yesterday.

6. Pick a corner of the cube. How many symmetries are there of the cube that send that corner to itself? Why do these form a subgroup of the group of group of symmetries of the cube? Pick a face of the cube instead of a corner and answer the same questions.

7. Pick a reflection of the cube. What is the subgroup of the symmetry group of the cube generated by it? Find the cosets of this group. Recover the order of symmetry cube again using this new information.

8. Pick a rotation of the cube. What is the subgroup of the symmetry group of the cube generated by it? Find the cosets of this group. Recover the order of symmetry cube again using this new information. OK. Maybe I'm done with ways to find the symmetry group of the cube. For now. Maybe.

9. Use our methods from above to find the size of the symmetry group of the tetrahedron and of the icosahedron (and hence the dodecahedron!), verifying what you found yesterday if you already found them.

What we've been doing in these last problems also relates to the Orbit-stabilizer theorem from group theory. It's about *group actions*. We say that a group G acts on a set S if we have a rule for elements of the group to send elements of S to elements of S where the group identity acts trivially and for any $g, h \in G$ and $s \in S$, $(gh)s = g(hs)$. We've already seen some group actions. For instance, the symmetry group on four objects has an action on a set of four objects. Also, given any group G and subgroup of it called H, there is an action of G on the set of cosets of H.

The orbit stabilizer theorem states that given a group action of G on a set S and any element $s \in S$, the order of G is equal to the product of the size of the *orbit* of s, which is the set of all elements of S where the group action could send s, and the size of the *stabilizer* of s, which is the set of all elements of G that send s to itself.

1. How can you phrase what you did during the earlier problems in terms of orbits and stabilizers?

2. Consider the group action of the permutation group on four objects on a set of four objects. Pick one of the objects. What is its orbit? Find all the elements in its stabilizer. Use these and the orbit-stabilizer theorem to recover the order of the permutation group on four objects.

3. Consider the group action of the permutation group on n objects on a set of four objects. Pick one of the objects. What is its orbit? Find all the elements in its stabilizer. Use these and the orbit-stabilizer theorem to recover the order of the permutation group on n objects.

2 More about generators

Yesterday we explored cyclic groups, which are finite groups generated by one element. But we can consider groups with more than one generator.

1. Consider the group of symmetries of the hexagon. Consider clockwise rotation by 120° and clockwise rotation by 180°. What happens when we compose these symmetries with themselves, in any possible order, repeatedly? Why does the set of all the symmetries we get this way form a subgroup of the symmetry group of the hexagon? This subgroup is said to be generated by these two elements.

2. Find a set of symmetries of the hexagon that generates the whole group. (Why does the set of all of the symmetries always work?). What's the smallest generating set that you can find? Describe all possible generating sets of this size.

3. Consider the subgroup of symmetries of the cube that fix one vertex. Find symmetries that generate this group. What's the least number of generators that you need?

4. Find generators for the entire group of symmetries of the cube. Can the group of symmetries of the cube be generated by just one element? Or just two?

5. Find generators for the groups of symmetries of the tetrahedron and icosahedron.

3 Isomorphisms

We call two groups *isomorphic* if there is a bijective correspondence between their elements that is compatible with the group structure. Effectively, isomorphic groups are the same group, though they often arise in different ways.

Consider the group of symmetries of the cube that fix one vertex and the group of rotations of the hexagon. Could they be isomorphic?

I claim that two cyclic groups of the same size are always isomorphic. Does this seem sensible? It's for this reason that we often talk about *the* cyclic group of a given order even though this cyclic group can arise in a lot of different ways.

4 More to do with cyclic groups

1. Yesterday we found cyclic groups by taking one element of a group, like one symmetry of a symmetry group, and composing it with itself repeatedly in order to eventually get a subgroup. What happens if we try to find the subgroup of the integers under addition generated by 1 in this way? What goes wrong? What could you do to fix the problem?

2. Consider the group generated by 1 under the operation addition mod 5. How many elements does it have? What is its identity? Which other elements of this group also generate it? Which elements of the group do not generate it?

3. Consider the group generated by 1 under the operation addition mod 12. How many elements does it have? Which other elements of this group also generate it? Which elements of the group do not generate it?

4. Based on the last two questions, can you form a guess for how many elements of the cyclic group of order n could possibly generate that group?

Some more cool examples of where cyclic groups arise are *roots of unity*, which live inside the complex plane.

1. Find all the solutions to the equation $x^2 = 1$, that is, all the "square roots of unity". Check that these form a group under multiplication.
2. Find all the solutions to the equation $x^3 = 1$. Check that these form a group under multiplication.
3. Find all the solutions to the equation $x^4 = 1$. Check that these form a group under multiplication.
4. Find all the solutions to the equation $x^6 = 1$. Check that these form a group under multiplication.
5. Graph the solutions to each of the previous problems. What do they look like?
6. Use the formula $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ to write your solutions to the previous questions as complex exponentials. What patterns do you notice?
7. Can you express what all the n-th roots of unity will look like in terms of complex exponentials?