Polytopes

1 Platonic Solids

Polyhedra are the 3-dimensional analogues of polygons. The regular convex polyhedra are called *Platonic solids*.

Although there are infinitely many regular polygons, there are finitely many Platonic solids. We will find all of them constructively, building solutions by analyzing how they look at a single vertex.

1.1 Triangles

We'll start with the smallest regular n-gon, an equilateral triangle, and see what Platonic solids we can build out of them.

1. Can we construct a Platonic solid with only two equilaterial triangles meeting at a vertex? What goes wrong? What does the number of edges meeting at a vertex tell us about the dimension fo the object?
2. What if three triangles meet at a vertex?
3. What if four triangles meet at a vertex?
4. What if five triangles meet at a vertex?
5. What if six triangles meet at a vertex?
6. How many Platonic solids can be built from equilateral triangles?

1.2 Squares

The next smallest regular n-gon is a square.

1. What if three squares meet at a vertex?
2. What if four squares meet at a vertex?
3. How many Platonic solids can be built from squares?

1.3 Pentagons

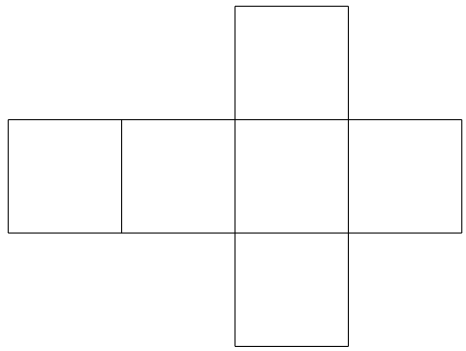
1. What if three pentagons meet at a vertex?
2. What if four pentagons meet at a vertex?
3. How many Platonic solids can be built from pentagons?

1.4 Hexagons

1. What if three hexagons meet at a vertex?
2. How many Platonic solids can be built from hexagons?
3. Can we build Platonic solics from n-gons for n greater than 6?

2 Nets

A *net* of a Platonic solid is exactly the information we need to make a Platonic solid out of a piece of paper. It has all the faces outlined, as well as information about how to glue them together. Here is a net for a cube:



1. What are the identifications you need to make on the net in order to glue it into a cube?

   Try different labels or using colors, etc.

2. Draw a net for a tetrahedron and indicate which identifications you need to make in order to glue it together. If you find yourself with some extra time, find nets for the other Platonic solids as well.

3 Euler characteristic

In the last section, we reasoned through what all possible Platonic solids could be. The Euler characteristic provides an alternate method to help find them, and is also interesting in other contexts.

The *Euler characteristic* of a Platonic solid is the number given by the following formula:

$$\# \text{ of vertices} - \# \text{ of edges} + \# \text{ of faces}$$

The *Euler characteristic* is a nice example of an invariant. An *invariant* is a simpler mathematical object that we assign to a more complicated mathematical object. In this case, we're assigning a number to a Platonic solid. Although we lose information from the more complicated object when we look at the invariant instead, the invariant can be easier to analyze.

1. What is the Euler characteristic of a cube?

2. What is the Euler characteristic of a tetrahedron?

3. Look at the picture of the net of the cube above. If you don't worry about identifying edges and vertices, what is its Euler characteristic?

4. Can you come up with an argument for why any such diagram for any Platonic solid will have the same Euler characteristic? It might be helpful to think about how these diagrams are built up, face by face.

5. The Euler characteristic we get from the net is different from the Euler characteristic of the folded up Platonic solid. How do the successive identifications affect the Euler characteristic? Can you use what you've found to make an argument for why the Euler characteristic of any Platonic solid will always be the same?

6. We now know that the Euler characteristic is always 2 for any Platonic solid, and can write this formula as follows:

$$V - E + F = 2$$

We can use this formula to help us find, in a different way than before, which Platonic solids are possible. Suppose we are trying to make a regular polyhedron made out of p-gons, where each vertex has q of them coming together at it. Why is the following formula also true?

$$qV = 2E = pF$$

7. Use the two formulas given in the last question to write formulas for V, E and F each in terms of p and q. Can you use these to find bounds for p and q? What are the possible pairs (p, q) allowed by our inequality?

4 Polytopes in four dimensions (and higher)

A more general term for a polygon or polyhedron or their analogues in higher dimensions is a *polytope*.

Now, suppose we're given a bunch of one type of Platonic solid, all the same size and asked to glue them together at their *faces* to make a shape that's closed, connected, and looks the same at each *edge*... in four dimensions...

4.1 Some history on the classification of regular polytopes in four dimensions: Alicia Boole Stott

The following information about Alicia Boole Stott comes from a short biography listed in the bibliography at the end:

Alicia was born in 1860 to George Boole, a mathematician, and Mary Everest Boole, a librarian and educator. Alicia's father died when she was only four years old, and her mother was forced to move her five daughters to London so that she could take a job as a librarian at Queens College, the first women's college in England. Alicia and her sisters were educated by their mother, but since the family was very poor, they had no chance for formal education. However, through her mother's library, Alicia was exposed to the academic circles of 19th century London whose members included names like Charles Darwin and H.G. Wells. One of the academics who Alicia spent time with was her brother-in-law, mathematician Charles Howard Hinton. At the time, Charles was working on a paper titled "What is the Fourth Dimension," and invited Alicia to help him analyze the problem with a set of little wooden cubes. Alicia proved to be a natural at understanding four dimensional geometry, and quickly surpassed Charles with her ability to visualize the fourth dimension. When she was just 18 years old, she coined the word "polytope," and discovered using only Euclidean methods the existence of the six regular convex 4-polytopes. Alicia continued her work in four dimensional geometry, and published several papers on it throughout her lifetime. She never did complete her formal education, but collaborated with mathematicians across Europe, including Pieter Schoute and H.S.M. Coxeter, and was awarded an honorary degree by the University of Groningen in 1914.

4.2 A start on classifying in four dimensions

We can apply the same arguments we used to find Platonic solids to find four-dimensional convex regular polytopes.

1. Find a net made out of cubes that describes the hypercube.

2. Can you find a range of numbers of tetrahedra that could be coming together at any given edge of a four-dimensional polytope made out of tetrahedra?

3. Take the smallest number from the last problem. Can you think of a way to glue together this minimal number of tetrahedra into a four-dimensional polytope that has the same number of tetrahedra coming together at every edge? If so, this is our first solution!

4. What about four-dimensional polytopes made out of other Platonic solids? What cases can you rule out using the above arguments?

And there we stop with our classification of regular, convex polytopes. However, here are some potential directions for further study:

- What are the Euler characteristics of polyhedra other than the Platonic solids? When is it not 2? What does it indicate?
- How might one generalize the Euler characteristic to four dimensions and use it to narrow down what four-dimensional polytopes made out of Platonic solids might be possible?
- After narrowing down the possibilities for what regular four-polyhedra *might* exist, to show they exist, they need to be constructed. One way of doing this is finding them inscribed in the regular four-polyhedra that we have already found. The alternating vertices of a cube form a tetrahedron, and the midpoints of the faces of a cube give the vertices of an octahedron. What regular four-polytopes can be found inscribed in a hypercube?
- In dimensions five and higher, there are always exactly three regular convex polytopes. How are they classified?
- Other potential topics of interest related to our work on polytopes: Coxeter diagrams, dualizing polyhedra, semiregular polyhedra

Bibliography

H. S. M. Coxeter has written multiple books with information about polytopes, including “Regular Polytopes” and “Introduction to Geometry”. His other books on geometry are also very cool!

Notes from some talks, lesson plans about Platonic solids by Anna Romanova (a recently graduated U of U grad student!) are linked on her website: https://www.math.utah.edu/ romanova

John Baez’s blog: https://johncarlosbaez.wordpress.com/

Alicia Boole Stott bibliography: http://www-history.mcs.st-andrews.ac.uk/Biographies/Stott.html