Plane Curves, Bézout's Theorem

1 Conic sections

1. Graph $z^2 = x^2 + y^2$. Find some points on it. What do the points on this look like where $z = 1$? Where $z = 2$? Where $z = -1$? Where $z = 0$?

2. Make a paper model approximating this structure. It will help with the next questions!

3. Find the intersections of $z^2 = x^2 + y^2$ with the following planes:

   (a) $z = 1$
   (b) $y = 1$
   (c) $z = x + 1$
   (d) $z = \frac{1}{2}x + 1$
   (e) $z = 0$
   (f) $y = 0$
   (g) $z = x$

   What do these intersections look like? I recommend drawing them on the figure you have produced.

2 Intersections

1. Where do the parabola $y = x^2 - 1$ and the x-axis intersect? Where do the parabola $y = x^2 - 2$ and the x-axis intersect? Where do the parabola $y = x^2$ and the x-axis intersect? Where do the parabola $y = x^2 + 1$ and the x-axis intersect? Graph these. What's happening?

2. In general, what happens when we take the intersection between the parabola $y = x^2 + b$ and the line $y = ax$? How many points of intersection are there? When are the solutions real? Graph some of these. What's happening?

3. Find all the points of intersection between the unit circle $x^2 + y^2 = 1$ and the following parabolas:

   (a) $y = x^2 + 2$
   (b) $y = x^2 - 1$
   (c) $y = x^2 + \frac{1}{4}$

   Draw some graphs of what's going on here. Does the number of points of intersection stay constant? How many points of intersection will the circle $x^2 + y^2 = 1$ have with any parabola of the form $y = x^2 + b$? How does the answer depend on b?

4. Find the intersections between the unit circle and the following ellipses:

   (a) $x^2 + 8y^2 = 4$
   (b) $x^2 + 4y^2 = 1$
   (c) $5x^2 + 6xy + 5y^2 + 6y = 5$
   (d) $4x^2 + y^2 + 6x = -2$

   How many are there with each? What do you notice about these?

5. How many points of intersection are there between the cubic $y = x^3 - x$ and the x-axis? How many points of intersection are there between the cubic $y = x^3$ and the line $y = 1$? Can you find an equation for a cubic and a horizontal line that have exactly two points of intersection?

6. How many points of intersection are there between the circle $x^2 + y^2 = 1$ and the following cubics:

   (a) $y = x^3$

   (b) $y = x^3 - x$

   (c) $y = 32x^3 - 8x$

   How many different numbers of intersections are there between the unit circle and different cubics? How does this change when you allow the coordinates of the intersections to be complex vs only real?