

BÉZOUT'S THEOREM:  
Intersection Multiplicity, Projective Space

## 1 Intersection multiplicity

In class on Friday, we saw some examples where plane curves intersect with multiplicity higher than one. This is a generalization of the idea of a polynomial having multiple roots.

Suppose  $P = 0$  and  $Q = 0$  are plane curves and  $p$  is a point in the plane. We will call the intersection multiplicity of  $P$  and  $Q$  at  $p$  by  $I_p(P, Q)$ . Intersection multiplicities can be calculated with the following rules:

R1.  $I_p(P, Q) = I_p(Q, P)$

R2.  $I_p(P, Q) = \infty$  if and only if  $P$  and  $Q$  have a common factor at  $p$

R3.  $I_p(P, Q) = 0$  if and only if  $p$  is not on both curves

R4.  $I_p(x, y) = 1$  where  $p = (0, 0)$

R5.  $I_p(P, Q_1Q_2) = I_p(P, Q_1) + I_p(P, Q_2)$

R6. If  $R$  is also a plane curve, then  $I_p(P, Q) = I_p(P + QR, Q)$

1. Graph the following sets of curves and compare them to one another. What's going on here?

(a)  $x = 0, y = 0, xy = 0$

(b)  $y - x^2 = 0, y^2 - x = 0, y^3 + x^3 - x^2y^2 - xy = 0$

(c)  $xy - 1 = 0, x = 0, x^2y - x = 0$

(d)  $x = 0, x = 0, x^2 = 0$

2. How does what you saw in the last problem shed some light on R2 and R5?

3. To shed some light on R6, compare the intersection of  $x = 0$  and  $y - 1 = 0$  with the following intersections:

(a) The intersection of  $x = 0$  and  $y - 1 + x^3 = 0$

(b) The intersection of  $x = 0$  and  $y - 1 + xy = 0$

(c) The intersection of  $y - 1 = 0$  and  $x + y^2 - y = 0$

(d) The intersection of  $y - 1 = 0$  and  $x + yx - x = 0$  (What does the second plane curve simplify to?)

What do these look like? Did the curves in all the examples always intersect in the same place? Did the intersections look like they could ever become tangent? Can you come up with plane curves  $P, Q$  and  $R$  where  $P$  and  $Q$  don't intersect at a tangent, but  $P + QR$  and  $Q$  do intersect at a tangent?

4. We're given that the multiplicity of the intersection of  $x = 0$  and  $y = 0$  at  $(0, 0)$  is 1. What about  $I_{(0,0)}(x, -y)$ ? Do you think it should also be 1, intuitively? Why does R5 tell us this is true?
5. Let's look at the multiplicity of the intersection of the parabola  $y = x^2$  with the  $x$ -axis at  $(0, 0)$ . The calculation is broken down into steps as follows:

$$I_{(0,0)}(y - x^2, y) = I_{(0,0)}(x^2, y) = I_{(0,0)}(x, y) + I_{(0,0)}(x, y) = 2$$

Compare what's been done with the rules above. What's been done in each of these steps?

6. As we discussed before, intersection multiplicity is a generalization of a polynomial having roots with multiplicity: whenever we compute the intersection multiplicity of the intersection of a polynomial with the  $x$ -axis at a point that is a root of the polynomial, the intersection multiplicity is equal to the multiplicity of that root. We checked that this is true for the parabola in the last problem. Check that the intersection multiplicity of  $y = x(x - 1)^3$  and  $y = 0$  at the point  $(1, 0)$  is 3.
7. Do the intersection multiplicities of some of the tangential intersections we saw on Friday behave how you would expect? For instance, what are the multiplicities of the following intersections?
  - (a) The intersection of  $x^2 + y^2 - 1$  and  $5x^2 + 6xy + 5y^2 + 6y - 5$  at  $(-1, 0)$
  - (b) The intersection of  $x^2 + y^2 - 1$  and  $4x^2 + y^2 + 6x + 2$  at  $(-1, 0)$

Are there any other intersection multiplicities from last time that you'd like to check?

## 2 The projective plane

We'll need to work inside the projective plane for our study of Bézout's theorem. The projective plane is not just two copies of the projective line. We have to construct it as follows: Start with three-dimensional space, but without the origin. Any two points that fall along the same line through the origin are considered to be the same. This is the projective plane! We write coordinates for the points in it in the form  $[x : y : z]$ .

1. Write down four other sets of coordinates for points that are considered to be equivalent to  $[1, 2, -1]$  in the projective plane.
2. Do the coordinates  $[1 : 0 : -1]$  and  $[1 : -1 : 0]$  represent the same point in projective space? Do the coordinates  $[2 : -1 : 1]$  and  $[2 : 1 : -2]$  represent the same point in projective space?
3. We think of any point  $(x, y)$  in the usual plane as being in the projective plane at  $[x : y : 1]$ . This rule gives us a function from the regular plane to the projective plane. Which points in the projective plane don't come from the ordinary plane?
4. If you haven't already, I highly recommend trying to make a drawing or a model of the projective plane.

5. In order to think of a plane curve in the ordinary plane as sitting inside the projective plane, we multiply every summand in it by a copy of  $z$  so that all of them have the same degree. For instance, we think of the parabola  $y - x^2 = 0$  in the plane as  $yz - x^2 = 0$  in the projective plane. Let's check that this makes some sense:
  - (a) Find a set of coordinates  $[x : y : z]$  that satisfies the equation  $yz - x^2 = 0$ . Do all the other points  $[a : b : c]$  that are equivalent to  $[x : y : z]$  also satisfy this equation?
  - (b) Why does the set of solutions to  $yz - x^2 = 0$  agree with  $y - x^2 = 0$  if we restrict to the ordinary plane?
6. Consider all the points in the projective plane that can be represented with a "1" in the first coordinate. What does this set of points look like? Which of these points are also in the "ordinary plane" in the sense we described in problem 3? Are there any points in projective space that aren't in this set of points or in the ordinary plane? What about points in the projective plane that have a "1" in the second coordinate?
7. In the last problem, we just described two planes other than the ordinary one that sit inside the projective plane. They all overlap with one another, but are different. What does the set of solutions of  $yz - x^2 = 0$  look like in each of those planes?
8. One of the interesting qualities that the projective plane has is that parallel lines actually always intersect. The equations  $x = 0$  and  $x - 1 = 0$  describe parallel lines in the ordinary plane. In the projective plane, their equations are  $x = 0$  and  $x - z = 0$ . Where do they intersect? Do they intersect at just one point?
9. At the end of the day on Friday, we thought about two parabolas side by side:  $y = x^2$  and  $y = (x - 1)^2$  that have just one point of intersection in the ordinary plane, with multiplicity one. What equations do these have in the projective plane? Do they intersect there?
10. What do the parabolas from the last problem look like in the  $xz$ -plane? What about the  $yz$ -plane? For each point of intersection, pick one of those planes and use the corresponding equations to calculate the intersection multiplicity. How many points of intersection does Bézout's Theorem tell us to expect, and is this example meeting that expectation?
11. Consider the concentric circles  $x^2 + y^2 = 1$  and  $x^2 + y^2 = 2$ . Can you verify Bézout's theorem and find four (with multiplicity) points of intersection? Are you curious about verifying Bézout's theorem for any other types of intersections we've seen?