Stereographic Projection

We talked about stereographic projection a little bit already, but there is a lot more fun stuff we can do with it.

1 Pythagorean triples

Pythagorean triples are whole numbers $a, b, c \in \mathbb{N}$ satisfying the equation $a^2 + b^2 = c^2$. The Pythagorean triple that is probably most commonly known is $3, 4, 5$.

A first few things we'll consider together: What changes about this problem if we allow a, b, c to be integers rather than just whole numbers? How can we think of these triples as corresponding to points on the unit circle with rational coordinates?

We'll need a slightly different version of stereographic projection than we introduced yesterday. We'll talk about this in lecture.

1. Our new type of stereographic projection sends points on the unit circle (minus the north pole) to the x-axis and vice versa.
   (a) Consider the points $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$, $(0, -1)$, $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ on the unit circle. Where on the x-axis does this stereographic projection send these points? Which point on the unit circle corresponds to the triple $3, 4, 5$ and what point on the x-axis does it get sent to?
   (b) Consider the following points on the x-axis: $(1, 0)$, $(2, 0)$, $(3, 0)$, $(\frac{1}{2}, 0)$. Where does the stereographic projection send them on the unit circle?
2. Consider any point on the x-axis with rational coordinates. We can write it as $(\frac{m}{n}, 0)$ for some m, n that are coprime. Where does on the circle does the stereographic projection send this point?
3. Suppose that (x, y) is a point on the unit circle with rational coordinates. Where on the x-axis does stereographic projection send this point?
4. Why do the last two questions show that we can find all possible Pythagorean triples by taking the stereographic projections of points on the x-axis with rational coordinates?
5. Use your answer to problem 2 to write the Pythagorean triple that we get from $(\frac{m}{n}, 0)$ and verify that is indeed a Pythagorean triple by plugging it into the formula $a^2 + b^2 = c^2$.
6. What Pythagorean triples correspond to $\frac{2}{3}$, $\frac{3}{5}$, $\frac{5}{3}$? Are they all primitive?
7. Show that if $m - n$ is odd, then the triple that $\frac{m}{n}$ corresponds to is primitive.
8. Are there infinitely many primitive Pythagorean triples?

Further questions one could ask: What are various qualities these Pythagorean triples could have (there's a long list of these on the wikipedia article, for instance). There are also similar equations to $a^2 + b^2 = c^2$ where there are interesting ways of trying to find out all possible whole number solutions.

2 More about stereographic projections

1. Consider six points on the circle at $0, \frac{2\pi}{6}, 2\frac{2\pi}{6}, \ldots$. Where does the stereographic projection map them to on the plane?
2. Consider the points on the x-axis at $(0, 0), (1, 0), (2, 0), (3, 0)$. Where does stereographic projection map them to on the circle?

Consider the stereographic projectios from a sphere to the plane.

1. What does the shadow cast by everything in the bottom half of the sphere look like? How big is it?

2. What does the shadow cast by everything in the top half of the sphere look like?

One fact that I find really surprising is that this stereographic projection both preserves angles (is conformal) and sends circles to circles, but doesn't preserve the centers of the circles. We'll explore this second property a bit together.

Given any symmetry of the sphere, we can consider it to correspond, via stereographic projection, to a map from the plane to itself.

1. What does rotation of the sphere around the vertical axis correspond to on the plane?

2. What does rotation of the sphere around a horizontal axis correspond to on the plane?

We can consider not just symmetries but other rigid motions of the sphere to correspond to maps of the plane via stereographic projection.

1. What does moving the sphere around horizontally correspond to on the plane?

2. If we move the sphere up one unit, how big is the shadow cast by its top half? If we move the sphere down by one unit instead, how big is the shadow cast by its bottom half?

If we restrict ourselves to the motions of the sphere given by translating and rotating, the corresponding maps on the plane are what's called Möbius transformations. If we consider the plane to be the complex plane, then the Möbius transformations are all of the form $f(z) = \frac{az+b}{cz+d}$ where $ad - bc \neq 0$.

For what values of a, b, c, d do the Möbius transformations correspond to translation and dilation? Which Möbius transformations correspond to rotating the sphere around a vertical axis? What about a horizontal axis?